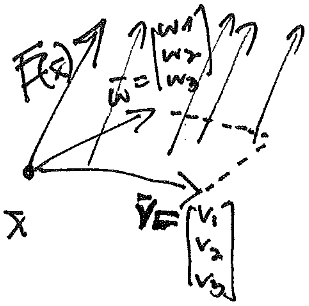
(95)

4/5/2017 EXAMPLE: $k=2, n=3$ (again anticipating §6.5) we'll see in a moment why we made this funny sign choice

Then $\varphi = +F_1\, dy\wedge dz - F_2\, dx\wedge dz + F_3\, dx\wedge dy$

for some vector field $\mathbb{R}^3 \xrightarrow{\bar F} \mathbb{R}^3$ that we will want to think of as a flux field (amount of fluid flow, or electric flux at each $\bar x \in \mathbb{R}^3$)

$$\bar x = \begin{pmatrix} x\\ y\\ z\end{pmatrix} \longmapsto \begin{bmatrix} F_1(\bar x)\\ F_2(\bar x)\\ F_3(\bar x)\end{bmatrix}$$



Note that a small parallelogram $P(\bar v, \bar w)$ anchored at $\bar x$ will have

$$\varphi(P(\bar v,\bar w)) = \left(F_1(\bar x)\, dy\wedge dz - F_2(\bar x)\, dx\wedge dz + F_3(\bar x)\, dx\wedge dy\right)\left(\begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}, \begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix}\right)$$

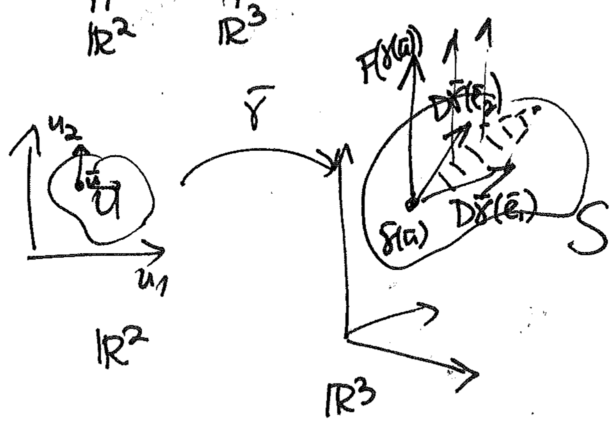
$$= \det\begin{bmatrix} F_1(\bar x) & v_1 & w_1\\ F_2(\bar x) & v_2 & w_2\\ F_3(\bar x) & v_3 & w_3\end{bmatrix} = \bar F(\bar x)\underset{\text{dot product}}{\bullet}(\bar v \underset{\text{cross product}}{\times} \bar w)$$

$= \mathrm{vol}_3\, P(\bar F(\bar x), \bar v, \bar w)$ "=" flow of $\bar F$ through parallelogram $P(\bar v,\bar w)$ in unit time

Thus one can reinterpret for a nicely parametrized surface $\begin{matrix} U & \xrightarrow{\bar\gamma} & S\\ \cap & & \cap\\ \mathbb{R}^2 & & \mathbb{R}^3\end{matrix}$

$$\int_S \varphi := \int_U \varphi\left(P_{\bar\gamma(\bar u)}(D\bar\gamma(\bar u))\right)|d^2\bar u|$$

$$= \int_U \underbrace{\mathrm{vol}_3\, P_{\bar\gamma(\bar u)}\left(\bar F(\bar x), D\bar\gamma(\bar u)(\bar e_1), D\bar\gamma(\bar u)(\bar e_2)\right)}_{\substack{\text{unit time flow of } \bar F\\ \text{through small parallelogram patch}\\ \text{on } S \text{ anchored at } \bar\gamma(\bar u)}} |d^2\bar u|$$



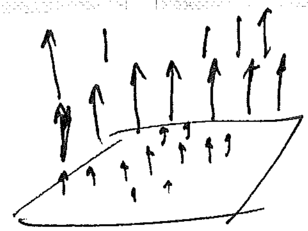
$$= \int_U \bar F(\bar x)\bullet\left(D\bar\gamma(\bar u)(\bar e_1) \times D\bar\gamma(\bar u)(\bar e_2)\right)|d^2\bar u|$$

called the flux integral of $\bar F$ ~~through~~ over S

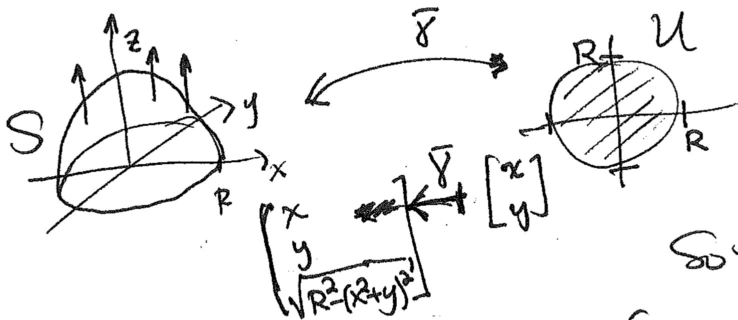
$= \underline{\text{flux}}$ of $\bar F$ $\underline{\text{through } S}$

$\rightsquigarrow$ $\boxed{\text{DEF'N 6.5.2: } \varphi = F_1\, dy\wedge dz - F_2\, dx\wedge dz + F_3\, dx\wedge dy \text{ is called the } \underline{\text{flux 2-form}}\ \Phi_{\bar F} \text{ of } \bar F}$

(96) e.g. for the flux field in $\mathbb{R}^3$ given by $F\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{bmatrix}0\\0\\z\end{bmatrix}$



what's the flux through the <u>upper</u> <u>hemisphere</u> S <u>of radius R</u>?



The flux 2-form $\Phi_{\vec{F}} = 0 \cdot dy\wedge dz - 0 \cdot dx\wedge dz + z\, dx\wedge dy = z\, dx\wedge dy$

$$D\bar{\gamma}\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\\ * & *\end{bmatrix}$$

(I don't care!)

So the flux integral is

$$\int_S \Phi_{\vec{F}} = \int_{\mathcal{U}} (z\, dx\wedge dy)\left(D\bar{\gamma}\begin{pmatrix}x\\y\end{pmatrix}(\vec{e}_1), D\bar{\gamma}\begin{pmatrix}x\\y\end{pmatrix}(\vec{e}_2)\right) |dx\, dy|$$

$$= \int_{\mathcal{U}} \sqrt{R^2-(x^2+y^2)}\, |dx\, dy|$$

$$\doteq \int_{x=-R}^{x=+R} \int_{y=-\sqrt{R^2-x^2}}^{y=+\sqrt{R^2-x^2}} \sqrt{R^2-(x^2+y^2)}\, dy\, dx$$

$(= \ldots\ldots$, let's not evaluate it!$)$

§6.3 Orientations of manifolds

- can be defined without parametrizations;
  1st let's do it on vector spaces...

<u>DEF'N 6.3.1</u>: For an n-dimensional (real) vector space V, an <u>orientation</u> of V is a map $\mathcal{B}_V := \{\text{ordered bases } (\vec{v}_1, \ldots, \vec{v}_n) \text{ for } V\} \xrightarrow{\Omega} \{\pm 1\}$

such that if $\bar{v} = (\vec{v}_1, \ldots, \vec{v}_n)$, $\bar{w} = (\vec{w}_1, \ldots, \vec{w}_n)$ have change-of-basis matrix $\Phi_{\bar{v}\to\bar{w}} = A$

then $\Omega(\bar{w}) = \underbrace{\mathrm{sgn}(\det A)}_{\pm 1} \cdot \Omega(\bar{v})$

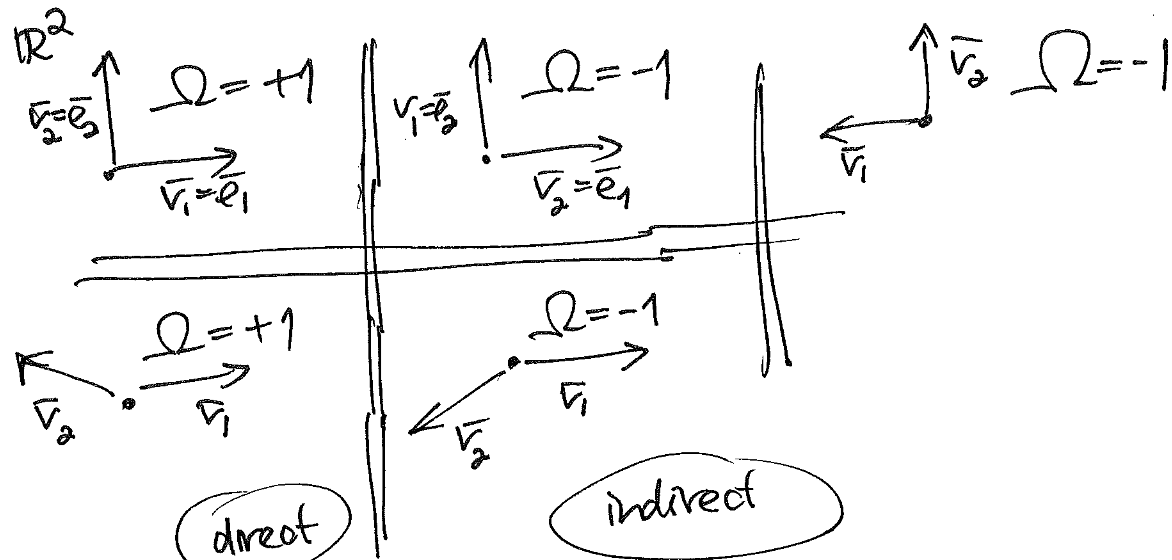
(97) This means there are only $\underline{2 \text{ orientations}}$ on V (!);

once you choose some $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n)$ to have $\Omega(\bar{v}) = +1$ (called $\bar{v}$ being $\underline{\text{direct}}$ for Ω;

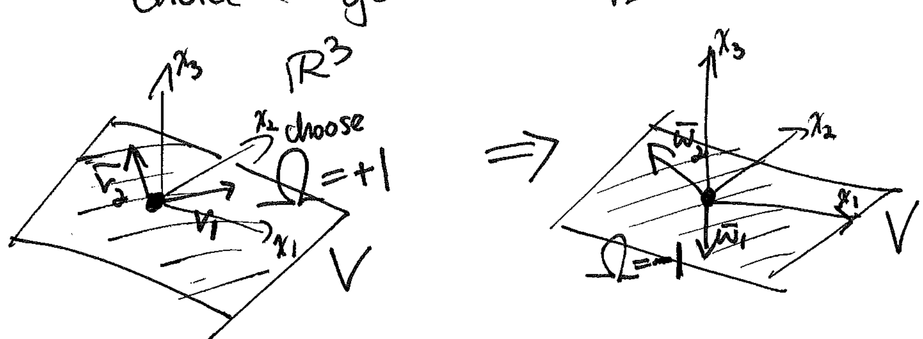
then all other $\Omega(\bar{w})$ are determined! $\Omega(\bar{w}) = -1$ is $\bar{w}$ being $\underline{\text{indirect}}$)

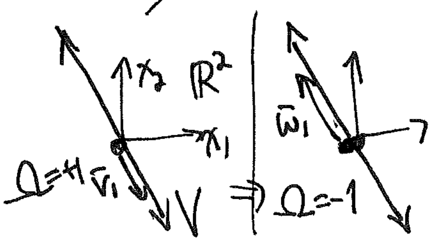
EXAMPLES:

① On $\mathbb{R}^n$, the $\underline{\text{standard}}$ orientation makes $\bar{v} = (\bar{e}_1, \dots, \bar{e}_n)$ $\underline{\text{direct}}$, i.e. $\Omega(\bar{v}) = +1$



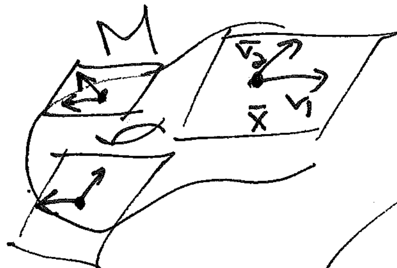
② On a proper subspace $V \subset \mathbb{R}^n$, there is no (natural) $\underline{\text{canonical}}$ choice – you need to $\underline{\text{pick}}$ one of the two.





On a manifold $M \subset \mathbb{R}^n$, one simply orients each tangent space $T_{\bar{x}}M$ in a continuous fashion:

DEF'N 6.3.3: For a k-dim'l manifold $M \subset \mathbb{R}^n$, an orientation is a continuous map



$$\mathcal{B}(M) := \left\{ (\bar{x}, \bar{v}_1, \dots, \bar{v}_k) : \bar{x} \in M, \; (\bar{v}_1, \dots, \bar{v}_k) \text{ an ordered basis of } T_{\bar{x}}M \right\} \xrightarrow{\Omega} \{\pm 1\}$$

$$\cap$$

$$\mathbb{R}^n \times \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k \text{ times}} = \mathbb{R}^{n(k+1)}$$

Think of this as the space of anchored parallelopipeds $P_{\bar{x}}(\bar{v}_1, \dots, \bar{v}_k)$

such that $\forall \bar{x} \in M$, the restriction of Ω to $\mathcal{B}_{\bar{x}}(M) = \left\{ (\bar{v}_1, \dots, \bar{v}_k) \text{ an ordered basis for } T_{\bar{x}}M \right\} \xrightarrow{\Omega} \{\pm 1\}$

is an orientation of $T_{\bar{x}}M$.

4/7/2017 >

EXAMPLES (PROP 6.3.4)

(0) $k=0$: A 0-dim'l manifold M is a finite set of points

$$M = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_t\}$$

and each $T_{\bar{x}_i}M$ is a 0-dim'l vector space, with only one basis, the empty list $()$ (!)

so you have to assign a $+1$ or -1 to each point $\bar{x}_i$ arbitrarily: