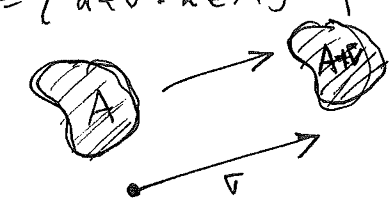
A few more reasonable properties of $\mathrm{vol}_n(-)$:

PROP 4.1.21, 4.1.22, 4.1.24: Assume $A, B \subset \mathbb{R}^n$ are pavable.

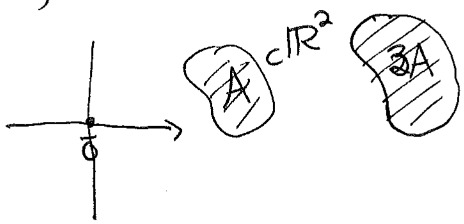
(i) If A, B disjoint $(A \cap B = \emptyset)$, then $A \cup B$ is pavable, and $\mathrm{vol}_n(A \cup B) = \mathrm{vol}_n A + \mathrm{vol}_n B$

(In fact, $A_1, \ldots, A_m$ disjoint, pavable $\Rightarrow \bigcup_{i=1}^m A_i$ pavable, $\mathrm{vol}_n\left(\bigcup_{i=1}^m A_i\right) = \sum_{i=1}^m \mathrm{vol}_n A_i$)

(ii) $\forall \bar{v} \in \mathbb{R}^n$, $A + \bar{v} := \{\bar{a} + \bar{v} : \bar{a} \in A\}$ is pavable, and $\mathrm{vol}_n(A + \bar{v}) = \mathrm{vol}_n(A)$



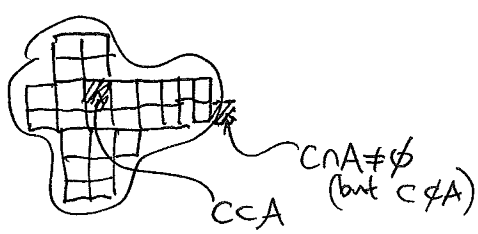
(iii) $\forall t \in \mathbb{R}$, $tA := \{t\bar{a} : \bar{a} \in A\}$ is pavable, and $\mathrm{vol}_n(tA) = t^n \mathrm{vol}_n(A)$



proof:

(i): If $A_1, \ldots, A_n$ are disjoint, then $1_{A_1 \cup \cdots \cup A_n} = 1_{A_1} + 1_{A_2} + \cdots + 1_{A_n}$ (so done by result on $\int f+g = \int f + \int g$).

(ii): For each N, $1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \subset A} C} \leq 1_A \leq 1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \cap A \neq \emptyset} C}$



and similarly $1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \subset A} C+\bar{v}} \leq 1_{A+\bar{v}} \leq 1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \cap A \neq \emptyset} C+\bar{v}}$

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$\Downarrow$

Because $\bigcup C+\bar{v}$ is a disjoint union of pavable sets $C+\bar{v}$, each a box, and $\mathrm{vol}_n(C+\bar{v}) = \mathrm{vol}_n C$

same reason as on left!

$$L\Big(1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \subset A} C+\bar{v}}\Big) \leq L(1_{A+\bar{v}}) \leq U(1_{A+\bar{v}}) \leq U\Big(1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \cap A \neq \emptyset} C+\bar{v}}\Big)$$

$$L\Big(1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \subset A} C}\Big) = L_N(1_A) \qquad\qquad U_N(1_A) = U\Big(1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \cap A \neq \emptyset} C}\Big)$$

$$L_N(1_A) \xrightarrow{N \to \infty} L(1_A) = \mathrm{vol}_n A = U(1_A) \xleftarrow{N \to \infty} U_N(1_A)$$

forces equality here!

(iii): Almost the exact same argument, starting with

$$\mathbb{1}_{\substack{\bigcup tC \\ C\in\mathcal{D}_N(\mathbb{R}^n): \\ C\subset A}} \leq \mathbb{1}_{tA} \leq \mathbb{1}_{\substack{\bigcup tC \\ C\in\mathcal{D}_N(\mathbb{R}^n): \\ C\cap A\neq\emptyset}}$$

and using $\mathrm{vol}_n(tC) = t^n \mathrm{vol}_n(C)$ for boxes C ∎
(or see book p.406)

At this stage, let's skip ahead a bit to §4.3 and see what functions are integrable.
At first, a not-so-useful sounding characterization:

THM 4.3.1: A function $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ bounded, of bounded support will be integrable $\iff$ $\forall \epsilon>0\ \exists N$ such that

$$\sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n): \\ \mathrm{osc}_C(f) > \epsilon}} \mathrm{vol}_n(C) < \epsilon$$

$\underbrace{\mathrm{osc}_C(f)}_{}$ the oscillation of f on C $\underset{\text{DEF 4.1.4}}{:=} M_C(f) - m_C(f)$ $\underset{\text{always}}{(\geq 0)}$

proof: ($\Rightarrow$): We'll show the contrapositive, i.e. assume $\exists\ \epsilon_0>0$ such that $\forall N$, $\displaystyle\sum_{\substack{C\in\mathcal{D}_N: \\ \mathrm{osc}_C(f)>\epsilon_0}} \mathrm{vol}_n(C) \geq \epsilon_0$. We'll prove f is not integrable, since $\forall N$ we'll have

$$U_N(f) - L_N(f) = \sum_{C\in\mathcal{D}_N} \big(M_C(f) - m_C(f)\big)\mathrm{vol}_n(C)$$

$$= \sum_{C\in\mathcal{D}_N} \underbrace{\mathrm{osc}_C(f)}_{\geq 0}\ \underbrace{\mathrm{vol}_n(C)}_{\geq 0}$$

$$= \sum_{\substack{C\in\mathcal{D}_N: \\ \mathrm{osc}_C(f)\leq\epsilon_0}} \mathrm{osc}_C(f)\,\mathrm{vol}_n(C) + \sum_{\substack{C\in\mathcal{D}_N: \\ \mathrm{osc}_C(f)>\epsilon_0}} \overbrace{\mathrm{osc}_C(f)}^{>\epsilon_0}\,\mathrm{vol}_n(C)$$

$$\geq 0 + \epsilon_0 \sum_{\substack{C\in\mathcal{D}_N: \\ \mathrm{osc}_C(f)\geq\epsilon_0}} \mathrm{vol}_n(C) \geq \epsilon_0\cdot\epsilon_0 = \epsilon_0^2$$

$$\Rightarrow \lim_{N\to\infty} U_N(f) - L_N(f) \neq 0.$$

($\Leftarrow$): Want to show, assuming the condition on the right, then f is integrable, i.e. $\lim_{N\to\infty} U_N(f) - L_N(f) = 0$. Given $\epsilon>0$, pick N as on right, so

$$U_N(f) - L_N(f) = \sum_{\substack{C\in D_N:\\ \mathrm{osc}(f)>\epsilon}} \mathrm{osc}_C(f)\,\mathrm{vol}_n(C) + \sum_{\substack{C\in D_N:\\ \mathrm{osc}(f)\le\epsilon}} \mathrm{osc}_C(f)\,\mathrm{vol}_n(C)$$

$$\le \sum_{\substack{C\in D_N:\\ \mathrm{osc}(f)>\epsilon}} M\cdot \mathrm{vol}_n(C) + \epsilon \sum_{\substack{C\in D_N:\\ 0<\mathrm{osc}(f)\le\epsilon}} \mathrm{vol}_n(C)$$

may as well assume this; forces $C\cap \mathrm{supp}(f)\neq\emptyset$

where $M := \left|\sup\{f(\bar{x}) : \bar{x}\in\mathbb{R}^n\} - \inf\{f(\bar{x}) : \bar{x}\in\mathbb{R}^n\}\right|$ a finite number, since f is bounded.

$$\le M \underbrace{\sum_{\substack{C\in D_N:\\ \mathrm{osc}(f)>\epsilon}} \mathrm{vol}_n(C)}_{<\epsilon} + \epsilon \underbrace{\left(\begin{array}{c}\text{any bound for}\\ \text{the sum of } \mathrm{vol}_n(C)\\ \text{for } C\cap\mathrm{supp}(f)\neq\emptyset\end{array}\right)}_{\substack{\text{finite, since } f\\ \text{has bounded}\\ \text{support}}}$$

M'

$$\le \left(\overset{M+M'}{\text{something finite and fixed}}\right)\cdot\epsilon$$

Hence $\lim_{N\to\infty} U_N(f) - L_N(f) = 0$ ∎

Although it didn't sound useful, we can actually apply this characterization to continuous functions with bounded support — they can't oscillate too much on small cubes because they're uniformly continuous

DEF'N 1.5.31: $\underset{\mathbb{R}^n}{\overset{\cap}{X}} \xrightarrow{f} \mathbb{R}$ is uniformly continuous (on X) if $\forall \epsilon>0\ \exists \delta>0$ with $|f(\bar{x})-f(\bar{y})|<\epsilon$ if $|\bar{x}-\bar{y}|<\delta$ $(\bar{x},\bar{y}\in X)$

THM 4.3.7: If $X\subset\mathbb{R}^n$ is compact (closed, bounded), then any continuous $X\xrightarrow{f}\mathbb{R}$ is uniformly continuous.

proof: Suppose not; then $\exists\epsilon_0>0$ and points $\bar{x}_i, \bar{y}_i\in X$ having $\lim_{i\to\infty} |\bar{x}_i - \bar{y}_i| = 0$ with $|f(\bar{x}_i) - f(\bar{y}_i)| \ge \epsilon_0$.

(44)

Since X is compact, by Bolzano-Weierstrass, can extract convergent subsequences $\bar{x}_{i_j} \longrightarrow \bar{a} \in X$

$\bar{y}_{i_j} \longrightarrow \bar{b} \in X$

But then $\lim_{i\to\infty} |\bar{x}_i - \bar{y}_i| = 0$ forces $\bar{a} = \bar{b}$

By continuity of f, $\exists J$ such that $j \geq J \Rightarrow$ $|f(\bar{x}_{i_j}) - f(\bar{a})| \leq \frac{\epsilon_0}{3}$
$|f(\bar{y}_{i_j}) - f(\bar{a})| \leq \frac{\epsilon_0}{3}$

hence $|f(\bar{x}_{i_j}) - f(\bar{y}_{i_j})| \leq |f(\bar{x}_{i_j}) - f(\bar{a})| + |f(\bar{a}) - f(\bar{y}_{i_j})| \leq \frac{\epsilon_0}{3} + \frac{\epsilon_0}{3} < \epsilon_0$, a contradiction. ∎

2/20/2017 This is the essence behind...

THM 4.3.6: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ continuous, with bounded support, is integrable.

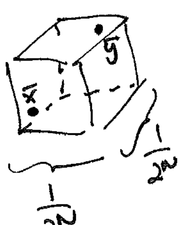
proof: Its (closed) support is closed, bounded, so compact,

Hence f is uniformly continuous (by THM 4.3.7 just proven), and so given $\epsilon > 0$, we can find $\delta > 0$ with

$|f(\bar{x}) - f(\bar{y})| < \epsilon$ whenever $|\bar{x} - \bar{y}| < \delta$.

Pick N large enough that $\frac{\sqrt{n}}{2^N} \leq \delta$, so whenever $\bar{x}, \bar{y} \in C$ a cube from $\mathcal{D}_N(\mathbb{R}^n)$, one has $|\bar{x} - \bar{y}| \leq \frac{\sqrt{n}}{2^N} < \delta$ and hence $|f(x) - f(y)| < \epsilon$, thus $\operatorname{osc}_C(f) < \epsilon$ for all cubes in $\mathcal{D}_N(\mathbb{R}^n)$ (!), i.e. $\sum_{\substack{C \in \mathcal{D}_N: \\ \operatorname{osc}(f) \geq \epsilon}} \operatorname{vol}_n C = 0 < \epsilon$ for sure. ∎

$\bar{x}, \bar{y} \in C \in \mathcal{D}_N(\mathbb{R}^n)$



$\Rightarrow |\bar{x} - \bar{y}| \leq \sqrt{\underbrace{\tfrac{1}{2^N} + \tfrac{1}{2^N} + \cdots + \tfrac{1}{2^N}}_{n \text{ times}}} = \frac{\sqrt{n}}{2^N}$

Better yet, f could have a few discontinuities, as promised earlier.

THM 4.3.10: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ (bounded with bounded support) which is continuous except on a (pavable) set of zero volume will always be integrable.