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§4.4 Measure zero

This is a concept for what sets $A \subset \mathbb{R}^n$ are negligible in integrals, that is much more flexible than our previous notion

$$\mathrm{vol}_n(A) = 0 \quad \text{i.e. } A \text{ pavable and } U_N(1_A) \to 0 \text{ as } N \to \infty$$

$$U_N(1_A) = \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n):\\ C \cap A \neq \emptyset}} \mathrm{vol}_n(C)$$

"A has content 0"
or volume

$\Updownarrow$

$\forall \epsilon > 0\ \exists$ (dyadic) cubes $C_1, \ldots, C_t$ covering $A \subset \bigcup_{i=1}^{t} C_i$ with total volume $\sum_{i=1}^{t} \mathrm{vol}_n(C_i) < \epsilon$

finitely many

DEF'N 4.4.1: $A \subset \mathbb{R}^n$ has measure 0 if $\forall \epsilon > 0\ \exists$ a countable sequence of boxes $B_i = [a_1,b_1] \times \cdots \times [a_n,b_n]$ with $A \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \mathrm{vol}(B_i) < \epsilon$

i.e. $\underbrace{B_1, B_2, B_3, \ldots}_{\text{allowed to shrink in size!}}$

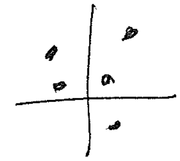
Goal of the section is a much better characterization of integrability:

THM 4.4.6: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ bounded values, support is integrable $\Leftrightarrow$ f is continuous except on a set of measure 0.

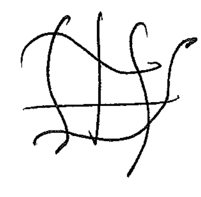
Before that, get some intuition for measure 0 from...

EXAMPLES: (1) If A has volume 0 (or content) then it has measure 0.

e.g. finite sets of points in $\mathbb{R}^n$



or finite union of graphs of continuous functions $\mathbb{R}^n \to \mathbb{R}$ in $\mathbb{R}^{n+1}$



② (THM 4.4.4) If $A_1, A_2, A_3, \ldots$ all have measure 0, then so does their union $\bigcup_{i=1}^{\infty} A_i$

proof: Given $\epsilon > 0$, find a countable union of boxes covering A_1 with total volume $\epsilon/2$ $(B_1^{(1)}, B_2^{(1)}, \ldots)$

A_2 — " — $\epsilon/2^2$ $(B_1^{(2)}, B_2^{(2)}, \ldots)$

A_3 — " — $\epsilon/2^3$ $(B_1^{(3)}, B_2^{(3)}, \ldots)$

$\vdots$

and then the union of all these boxes covers $\bigcup_{i=1}^{\infty} A_i$, with total volume $\leq \frac{\epsilon}{2} + \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \ldots = \epsilon$

(and it's still a countable list of boxes:

$$\begin{array}{cccc} \vdots & \vdots & & \\ B_1^{(3)} & B_2^{(3)} & & \\ B_1^{(2)} & B_2^{(2)} & B_3^{(2)} & \cdots \\ B_1^{(1)} & B_2^{(1)} & B_3^{(1)} & \cdots \end{array}$$

) ∎

③ In particular, $A = \mathbb{Q} \cap [0,1]$ was not of volume 0 (not even pavable, since 1_A was not integrable)

but it has measure 0, since $\mathbb{Q} \cap [0,1]$ is countably many points:

$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$

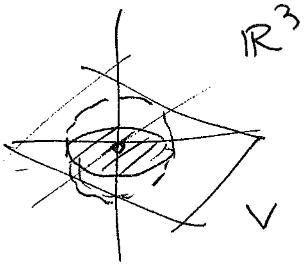
repeats, but who cares?

④ In fact, $A = \mathbb{Q} \subset \mathbb{R}^1$ is also of measure 0, since $\mathbb{Q}$ is countable

$$\begin{array}{cccccccc} \vdots & & & & & & & \\ \frac{0}{3} & \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} & \frac{-2}{3} & & & \\ \frac{0}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{2}{2} & \frac{-2}{2} & & & \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \end{array}$$

So measure 0 doesn't even require A bounded!

⑤ Once one believes that bounded subsets of proper linear subspaces $V \subset \mathbb{R}^n$, say $V \cap \underbrace{B_r(\bar{0})}_{\text{ball of radius } r \text{ about } \bar{0}}$, are of volume 0

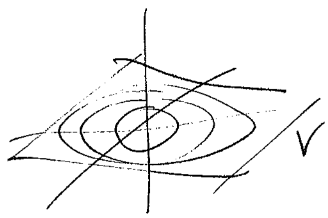


(which we skipped the proof of PROP 4.3.5 about graphs of cont. functions having vol 0)

then this gives

<u>PROP 4.4.c</u>: A proper linear subspace $V \subset \mathbb{R}^n$ has measure 0

$$\text{proof: } V = \underbrace{\bigcup_{i=1}^{\infty} \underbrace{V \cap B_i(\bar{0})}_{\substack{\text{volume 0,} \\ \text{so measure 0}}}}_{\text{so measure 0}} \blacksquare$$



picture

Let's now prove (½ of)

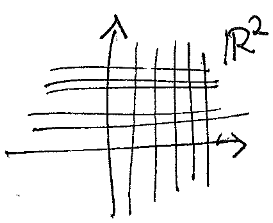
<u>THM 4.4.6</u>: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ bounded values, support is integrable $\iff$ f continuous except on a set of measure 0.

proof: ($\Leftarrow$) is very similar in spirit to proof of THM 4.3.10; can read it in book pp 435-436.

($\Rightarrow$) is not explained convincingly (to my tastes) in book, so let's rephrase it. Assume f is integrable, and let

$$\Delta := \{\bar{x}_0 \in \mathbb{R}^n : f \text{ is } \underline{\text{dis}}\text{continuous at } \bar{x}_0\}$$

$$= \underbrace{\{\bar{x}_0 \in \mathbb{R}^n \text{ having at least one coordinate in } \mathbb{Q}\} \cap \Delta}_{\substack{\text{a set of measure zero,} \\ \text{since it's a union of countably} \\ \text{many } (n-1)\text{-dimensional subspaces}}} \cup \underbrace{\{\bar{x}_0 \in \mathbb{R}^n \text{ with no coordinates in } \mathbb{Q}\} \cap \Delta}_{\substack{\text{only need to show this} \\ \text{set has measure 0;} \\ \bar{x}_0 \text{ in this set are in} \\ \text{the interior of every} \\ \text{cube } C \in \mathcal{D}_N(\mathbb{R}^n)\ \forall N}}$$



$$= \qquad A \qquad \cup \qquad B$$

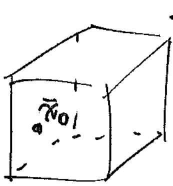
Note that if we fix some $\epsilon > 0$,

$$\text{then } B = \bigcup_{k=1}^{\infty} \underbrace{\left\{\bar{x}_0 \in \mathbb{R}^n \text{ with no coords in } \mathbb{Q} : \exists\, \bar{y}_1, \bar{y}_2, \ldots \text{ converging to } \bar{x}_0 \text{ with } |f(\bar{y}_i) - f(\bar{x}_0)| > \frac{\epsilon}{2^k}\right\}}_{\text{Call this set } B_k}$$

~~[illegible]~~

$\Downarrow$

$\bar{x}_0$ lives in a cube $C \in \mathcal{D}_N(\mathbb{R}^n)$ with $\text{osc}_C(f) > \frac{\epsilon}{2^k}$



$C \in \mathcal{D}_N(\mathbb{R}^n)$

For each k, integrability of f implies we can pick $N_k > 0$ so that

$$\sum_{\substack{C \in \mathcal{D}_{N_k}(\mathbb{R}^n): \\ \mathrm{osc}_C(f) > \frac{\epsilon}{2^k}}} \mathrm{vol}_n(C) < \frac{\epsilon}{2^k}$$

Then $$B = \bigcup_{k=1}^{\infty} B_k \subset \underbrace{\bigcup_{k=1}^{\infty} \underbrace{\bigcup_{\substack{C \in \mathcal{D}_{N_k}(\mathbb{R}^n): \\ \mathrm{osc}_C(f) > \frac{\epsilon}{2^k}}}}_{\text{finite}} C}_{\text{a countable union of cubes of volume } \leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.}$$

COR 4.4.10 : 4.4.11

If $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$, $\mathbb{R}^n \xrightarrow{g} \mathbb{R}$ are both integrable, then so is $f \cdot g$,

and so is
$$\begin{aligned} \mathbb{R}^n &\xrightarrow{h} \mathbb{R} \\ \bar{x} &\longmapsto f(\bar{x} - \bar{v}_0) \end{aligned}$$
for any fixed $\bar{v}_0 \in \mathbb{R}^n$

proof: If $\Delta_f := \{\bar{x}_0 \in \mathbb{R}^n : f \text{ is discontinuous at } f\}$

then $$\underbrace{\Delta_{fg} \subset \underbrace{\Delta_f}_{\text{meas. } 0} \cup \underbrace{\Delta_g}_{\text{meas } 0}}_{\Rightarrow \text{ meas } 0}$$, so fg is integrable.

Also $$\Delta_h = \underbrace{\bar{v}_0 + \underbrace{\Delta_f}_{\text{meas } 0}}_{\Rightarrow \text{ meas } 0}$$, so h is integrable. ∎