(1)

Math 3593H Spring 2017

Office hours? MW 9:05
Tues 1:25

OVERVIEW

Chap 3 – approximating manifolds, linearly (tangent space)
quadratically (analyze max/mins, curvature) $\boxed{\text{Math 3378}}$
via higher degree multivariate polynomials (Taylor polynomials)

Chap 4 – integrating functions over $\mathbb{R}^n$; mostly Riemannian, Lebesgue in §4.11 $\boxed{\text{Math 5615}}$

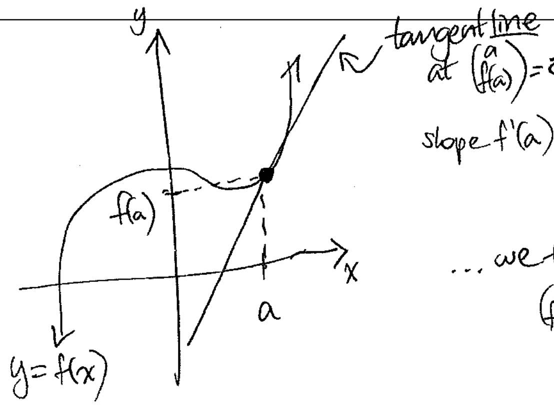
Chap 5 – arc length, surface area, higher volume via integration

Chap 6 – integration with orientations considered
– differential forms (divergence, Green, Stokes's Thms)

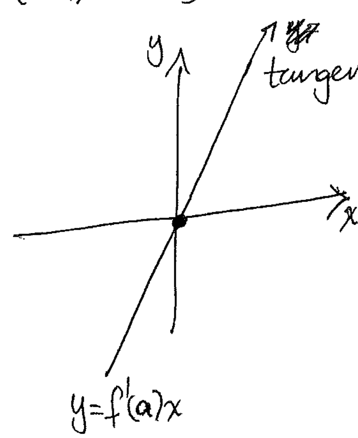
---

§3.2 Tangent spaces

As we've said in one variable, for the graph $M = \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} : x \in \mathbb{R} \right\}$ of $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ near $\begin{pmatrix} a \\ f(a) \end{pmatrix} = c$ ...



... we translate $\begin{pmatrix} a \\ f(a) \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



tangent space, a linear subspace of $\mathbb{R}^2$
= the graph of $f'(a) : \mathbb{R} \to \mathbb{R}$, $x \mapsto f'(a)x$
$= \left\{ \begin{pmatrix} x \\ f'(a)x \end{pmatrix} : x \in \mathbb{R} \right\}$

DEF'N (3.2.1) For $\bar{f} : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and M its graph $\left\{ \begin{pmatrix} \bar{x} \\ \bar{f}(\bar{x}) \end{pmatrix} : \bar{x} \in \mathbb{R}^k \right\} \subset \mathbb{R}^n$,
define the tangent space $T_{\begin{pmatrix} \bar{a} \\ \bar{f}(\bar{a}) \end{pmatrix}} M :=$ the graph $\left\{ \begin{pmatrix} \bar{x} \\ D\bar{f}(\bar{a})\bar{x} \end{pmatrix} : \bar{x} \in \mathbb{R}^k \right\} \subset \mathbb{R}^n$
of $D\bar{f}(\bar{a}) : \mathbb{R}^k \to \mathbb{R}^{n-k}$

(2)

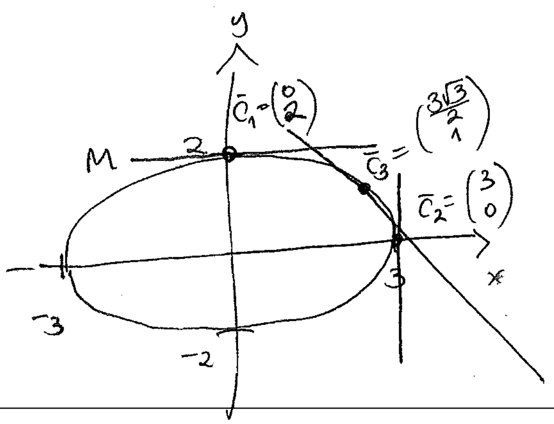
... and since for every k-dimensional manifold $M \subset \mathbb{R}^n$, by definition one can express it locally (by re-indexing the coordinates) around some $\bar{c} \in M$, $\bar{c} = \binom{\bar a}{f(\bar a)}$, as the graph $\left\{ \binom{\bar x}{f(\bar x)} : \bar x \in U \right\}$ for some $U \xrightarrow{f} \mathbb{R}^{n-k}$, $U \subset \mathbb{R}^k$,

define the tangent space $T_{\bar c}M :=$ the graph $\left\{ \binom{\bar x}{Df(\bar a)\bar x} : \bar x \in \mathbb{R}^k \right\}$

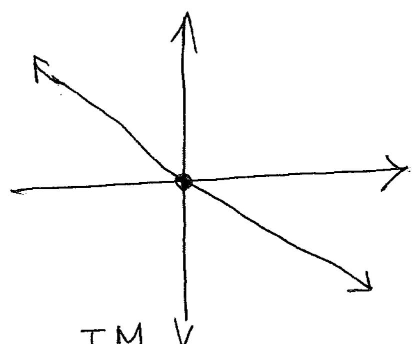
of $\mathbb{R}^k \xrightarrow{Df(\bar a)} \mathbb{R}^{n-k}$

EXAMPLE: The ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is a manifold $M \subset \mathbb{R}^2$

and here are 3 of its tangent lines & tangent spaces:



$\rightsquigarrow$



$T_{\bar c_1}M =$ x-axis
$y = 0 \cdot x$

use
$y = +2\sqrt{1 - \frac{x^2}{9}}$

$\frac{dy}{dx} = \frac{1}{2} \cdot \left(\frac{-2x}{9}\right) 2 \left(1 - \frac{x^2}{9}\right)^{-\frac{1}{2}}$

$\left.\frac{dy}{dx}\right|_{\binom{0}{2}} = 0 \Rightarrow$ graph of $\boxed{y = 0 \cdot x}$

$T_{\bar c_2}M$ ‖ y-axis
$x = 0 \cdot y$

use
$x = +3\sqrt{1 - \frac{y^2}{4}}$

$\frac{dx}{dy} = \frac{1}{2}\left(\frac{-2y}{4}\right) 3 \left(1 - \frac{y^2}{4}\right)^{-\frac{1}{2}}$

$\left.\frac{dx}{dy}\right|_{\binom{3}{0}} = 0 \Rightarrow$ graph of $\boxed{x = 0 \cdot y}$

For $T_{\bar c_3}M$, could use either parametrization to get the tangent space

e.g. $y = \pm 2\sqrt{1 - \frac{x^2}{9}}$ OR $x = +3\sqrt{1 - \frac{y^2}{4}}$

$\frac{dy}{dx} = \frac{-2x}{9}\left(1 - \frac{x^2}{9}\right)^{-\frac{1}{2}}$

$\left.\frac{dy}{dx}\right|_{\binom{3\sqrt3/2}{1}} = \frac{-6\sqrt3}{18}\left(1 - \frac{3}{4}\right)^{-\frac{1}{2}} = \frac{-\sqrt3}{3}\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \frac{2}{-\sqrt3} \Rightarrow$ graph of $\boxed{y = \frac{2}{-\sqrt3}x}$

$\frac{dx}{dy} = -\frac{3y}{4}\left(1 - \frac{y^2}{4}\right)^{-1/2}$

$\left.\frac{dx}{dy}\right|_{\binom{3\sqrt3/2}{1}} = -\frac{3}{4}\left(1 - \frac{1}{4}\right)^{-1/2} = \frac{-\sqrt3}{2} \Rightarrow$ graph of $\boxed{x = \frac{-\sqrt3}{2}y}$

(3)

How do we know <u>either</u> local parametrization should work there?

A more "parametrization neutral" formulation of $T_{\bar{c}}M$ when M is defined <u>implicitly</u>...

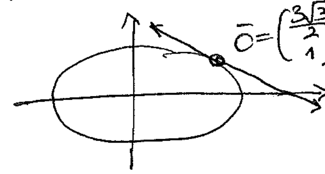
<u>THM 3.2.4</u>: If $\bar{c}$ is a point on a k-dimensional manifold $M \subset \mathbb{R}^n$ defined locally by $\bar{F}(\bar{z}) = \bar{0}$ for some $\underset{\mathbb{R}^n}{\overset{U}{\cap}} \xrightarrow{\bar{F}} \mathbb{R}^{n-k}$ with $D\bar{F}(\bar{c})$ of full rank $n-k$

then $T_{\bar{c}}M = \ker D\bar{F}(\bar{c}) \quad (\subset \mathbb{R}^n)$.

<u>EXAMPLES</u>:

(1) $M = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} = 1 \right\} = \left\{ \bar{F}(\bar{z}) = \bar{0} \right\}$

for $F\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^2}{9} + \frac{y^2}{4} - 1$, $\mathbb{R}^2 \longrightarrow \mathbb{R}$



$\bar{c} = \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ 1 \end{pmatrix}$

$D\bar{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{2x}{9} & \frac{y}{2} \end{bmatrix}$, $\mathbb{R}^2 \longrightarrow \mathbb{R}$

so $D\bar{F}\begin{pmatrix} \frac{3\sqrt{3}}{2} \\ 1 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2} \end{bmatrix}$, $\mathbb{R}^2 \longrightarrow \mathbb{R}$

which has $\ker D\bar{F}(\bar{c}) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \frac{y}{\sqrt{3}} + \frac{x}{2} = 0 \right\}$

$= \left\{ y = \frac{-\sqrt{3}}{2} x \right\}$

$= \left\{ x = \frac{-2}{\sqrt{3}} y \right\}$

(2) A surface $M \subset \mathbb{R}^3$ defined by $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

will have tangent space $T_{\underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}_{\bar{c}}} M$ given by $\ker DF\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \ker \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{bmatrix}\Bigg|_{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}$

$= \Bigg(\underbrace{\nabla F\Big|_{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}}_{\text{gradient vector for } F \text{ at } \bar{c}} \Bigg)^{\perp}$