(4)

<u>proof of THM 3.2.4:</u>

We'll actually prove a bit more : PROP 3.2.7

and tie up a loose end about Imp. Function Thm.

Since $\underset{\left(\begin{smallmatrix}\bar{a}\\ f(\bar{a})\end{smallmatrix}\right)}{\bar{c}} \in M = \{ \bar{F}(\bar{z}) = \bar{0} \}$ has $\mathbf{D}\bar{F}(\bar{c})$ of full rank $n-k$, $\mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$

we can re-index variables so
$$D\bar{F}(\bar{c}) = \begin{array}{c} \\ n-k\Big\{ \end{array} \begin{array}{c} \overbrace{\text{non-pivotal } x_1,\ldots,x_k} \quad \overbrace{\text{pivotal } y_1,\ldots,y_{n-k}} \\ \left[\begin{array}{c|c} Q & \underset{\text{invertible}}{P} \end{array}\right] \end{array}$$

and then the Implicit Function Thm says M is locally the graph of some parametrization γ:

$$\begin{array}{ccccc}
V^{\text{open}} & \xrightarrow{\gamma} & \mathcal{U}^{\text{open}} & \xrightarrow{\bar{F}} & \mathbb{R}^{n-k} \\
\cap & & \cap & & \\
\mathbb{R}^k & & \mathbb{R}^n & & \\
\bar{x} & \longmapsto & \begin{pmatrix}\bar{x}\\ f(\bar{x})\end{pmatrix} & & \\
& & \bar{z} = \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} & \longmapsto & \bar{F}(\bar{z})
\end{array}$$

Since $\bar{F}\circ\gamma(\bar{x}) = \bar{F}\begin{pmatrix}\bar{x}\\ \gamma(\bar{x})\end{pmatrix} = \bar{0}$, taking derivatives at $\bar{x} = \bar{a}$ and chain rule gives

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, "D\gamma(\bar{a})"'] \arrow[rr, bend left, "\bar{0}"] & \mathbb{R}^n \arrow[r, "D\bar{F}(\bar{c})"'] & \mathbb{R}^{n-k}
\end{tikzcd}$$

$$D\gamma(\bar{a}) = \begin{array}{c} k\{ \\ n-k\{ \end{array} \underbrace{\begin{bmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline D\bar{f}(\bar{a}) \end{bmatrix}}_{k} \qquad D\bar{F}(\bar{c}) = [\underbrace{Q}_{k} \mid \underbrace{P}_{n-k}]\}{n-k}$$

i.e. $[D\bar{F}(\bar{c})][D\gamma(\bar{a})] = \bar{0}_{(n-k)\times k}$

$$n-k\{[Q \mid P] \underbrace{\begin{bmatrix} I_k \\ \hline D\bar{f}(\bar{a}) \end{bmatrix}}_{k} = \bar{0}$$

$$\Rightarrow Q + P\, D\bar{f}(\bar{a}) = \bar{0}$$

i.e. $$\boxed{\begin{aligned} D\bar{f}(\bar{a}) &= -P^{-1}Q \\ &= -\left[D\bar{F}(\bar{c})\Big|_{\substack{\text{pivot cols}\\ y_1,\ldots,y_{n-k}}} \right]^{-1} \left[D\bar{F}(\bar{c})\Big|_{\substack{\text{non-pivot cols}\\ x_1,\ldots,x_k}} \right] \end{aligned}}$$

"Implicit Differentiation"

(5) Also, by definition

$$T_{\bar{c}}M = \text{graph } \left\{ \begin{pmatrix} \bar{x} \\ Df(\bar{a})\bar{x} \end{pmatrix} : \bar{x} \in \mathbb{R}^k \right\} \text{ of } \mathbb{R}^k \xrightarrow{Df(\bar{a})} \mathbb{R}^{n-k}$$

$$\begin{pmatrix} \bar{x} \\ Df(\bar{a})\bar{x} \end{pmatrix} = D\gamma(\bar{a})\bar{x}$$

i.e. $\boxed{T_{\bar{c}}M = \operatorname{im} D\gamma(\bar{a})}$ ⟵ Roughly, PROP 3.2.7

Lastly, $[DF(\bar{c})][D\gamma(\bar{a})] = \bar{0}$ shows

$$(T_{\bar{c}}M) \quad \underbrace{\operatorname{im} D\gamma(\bar{a})}_{\substack{\text{has dimension} \\ \operatorname{rank} D\gamma(\bar{a}) \\ = \operatorname{rank}\begin{bmatrix} I_k \\ * \end{bmatrix} = k}} \subset \underbrace{\ker DF(\bar{c})}_{\substack{\text{has dimension} \\ \underbrace{n - \underbrace{\operatorname{rank} DF(\bar{c})}_{n-k}}_{k}}}$$

by rank-nullity formula

$$\Rightarrow \quad \underbrace{\operatorname{im} D\gamma(\bar{a})}_{= T_{\bar{c}}M} = \ker DF(\bar{c})$$, as desired ∎

§3.3 Multivariate Taylor polynomials

Recall in one variable, we approximated $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$, $x \longmapsto f(x)$

near $x = a$ with its k^{th} order Taylor polynomial

$$P^k_{f,a}(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

or ($x = a+h$, $h = x-a$)

$$P^k_{f,a}(a+h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{2!}h^2 + \dots + \frac{f^{(k)}(a)}{k!}h^k = \sum_{m=0}^{k} \frac{f^{(m)}(a)}{m!}h^m$$

EXAMPLE: $f(x) = \sin(x)$ near $x = 0$

$$\approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$P^1 = P^2$, $P^3 = P^4$, $P^5 = P^6$ etc.

(6) The point of the $\frac{1}{m!}$ was to make $P^k_{f,a}(x)$ have <u>same</u> <u>derivatives</u> at $x=a$ as $f(x)$ up through k^{th} derivative.

In several variables, say $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

$$\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto f(\bar{x})$$

has many higher partial derivatives. How to index them?

<u>EXAMPLE:</u>

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \sin(x^2+y)$$

$\frac{\partial}{\partial x}$:
$$\frac{\partial f}{\partial x} = 2x\cos(x^2+y)$$

$\frac{\partial}{\partial y}$:
$$\frac{\partial f}{\partial y} = \cos(x^2+y)$$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2+y) - 4x^2\sin(x^2+y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = -2x\sin(x^2+y) = \frac{\partial^2 f}{\partial x \partial y}$$

SAME!

$$\frac{\partial^2 f}{\partial y^2} = -\sin(x^2+y)$$

$$\frac{\partial^3 f}{\partial x^3} = -8x^3\cos(x^2+y) - 12x\sin(x^2+y)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2} = -4x^2\cos(x^2+y)$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial^3 f}{\partial y^2 \partial x} = -2x\cos(x^2+y)$$

$$\frac{\partial^3 f}{\partial y^3} = -\cos(x^2+y)$$

<u>THM 3.3.9 + COR 3.3.11</u>: If $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ has each $\frac{\partial f}{\partial x_i}$ differentiable, then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ $\forall i,j$

If it has each $(k-1)^{st}$ order partial derivative differentiable, then each k^{th} order partial derivative is independent of order of $\frac{\partial}{\partial x_i}$'s.

<u>Assuming this</u> for the moment, define $\underbrace{D_{(i_1,i_2,\dots,i_n)}}_{I} f := \frac{\partial^k f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = D_I f$ where $k := i_1 + \dots + i_n$

(actually, let's not do the proof in lecture; see proof in book Appendix A.9, or Bubakers Lecture 31 showing $\frac{\partial^2 f}{\partial x_j \partial x_i} = \lim_{t\to 0} \frac{1}{t^2}\left(f(\bar a + t\bar e_i + t\bar e_j) - f(\bar a + t\bar e_j) - f(\bar a + t\bar e_i) + f(\bar a)\right)$, which is <u>symmetric</u> in i & j)