(1)

Some linear algebra/determinant loose ends

Recall for A $n\times n$,

$$\det A := \sum_{\sigma \in \mathrm{Perm}_n} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

<u>PROP</u> (Laplace expansion along a column)

For each $j=1,2,\ldots,n$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det\left(A \setminus \left\{\begin{matrix}\text{row } i\\ \text{col } j\end{matrix}\right\}\right)$$

EX<u>AMPLE</u>:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)^{1+2} a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + (-1)^{2+2} a_{22} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$+ (-1)^{3+2} a_{32} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

expand along column $j=2$

<u>proof</u>: (sketch) WLOG can assume $j=1$ using fact that $\det A$ is negated when one swaps columns of A.

So want to show $\det A \overset{?}{=} \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det\left(A \setminus \left\{\begin{matrix}\text{row } i\\ \text{col } 1\end{matrix}\right\}\right)$

$\parallel$

$$\sum_{i=1}^{n} \sum_{\substack{\sigma \in \mathrm{Perm}_n: \\ \sigma(i)=1}} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)} \cdots a_{i,\underset{1}{\underset{\parallel}{\sigma(i)}}} \cdots a_{n,\sigma(n)}$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \underbrace{\sum_{\substack{\sigma \in \mathrm{Perm}_n: \\ \sigma(i)=1}} (-1)^{i+1} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)} \cdots \widehat{a_{i,\sigma(i)}} \cdots a_{n,\sigma(n)}}_{\underset{\text{EXERCISE}}{=} \det\left(A \setminus \left\{\begin{matrix}\text{row } i\\ \text{col } 1\end{matrix}\right\}\right)}$$

(2)

COR: If we define the adjugate matrix of the $n\times n$ matrix A

as $$(\operatorname{adj} A)_{i,j} := (-1)^{i+j} \det\left(A \setminus \begin{Bmatrix}\text{row } i,\\ \text{col } j\end{Bmatrix}\right)$$

then $(\operatorname{adj} A)^T \cdot A = \det A \cdot I_n$.

Hence if A is invertible, then $\boxed{A^{-1} = \frac{1}{\det A} (\operatorname{adj} A)^T}$

EXAMPLE: $n=2$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has $\operatorname{adj} A = \begin{bmatrix} +d & -c \\ -b & +a \end{bmatrix}$

so $$(\operatorname{adj} A)^T \cdot A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

proof of COR: $$\left((\operatorname{adj} A)^T \cdot A\right)_{i,k} = \sum_{j=1}^{n} (\operatorname{adj} A)_{ij} \cdot a_{jk}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} \det\left(A \setminus \begin{Bmatrix}\text{row } j,\\ \text{col } i\end{Bmatrix}\right) \cdot a_{jk}$$

Laplace expansion along column i

$$= \det\left(A \text{ with col } i \text{ replaced by col } k\right)$$

$$= \begin{cases} \det A & \text{if } i=k \\ \det\left(A \text{ with col } k \text{ repeated as col } i\right) = 0 & \text{if } i \neq k \end{cases}$$

i.e. $$(\operatorname{adj} A)^T \cdot A = \begin{bmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{bmatrix} \qquad \blacksquare$$

(3)

A closely related COR:

COR (Cramer's rule) (EXER. 4.8.18):

When A is $n \times n$ invertible, the solution $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ to $A\bar{x} = \bar{b}$ has
$$x_j = \frac{\det(A \text{ with col } j \text{ replaced by } \bar{b})}{\det(A)}$$

proof: $\bar{x} = A^{-1}\bar{b}$

so
$$x_j = (A^{-1}\bar{b})_j = \frac{1}{\det A}\left[(\operatorname{adj} A)^T \bar{b}\right]_j = \frac{1}{\det A}\sum_{i=1}^{n} (\operatorname{adj} A)^T_{ji}\, b_i$$
$$= \frac{1}{\det A}\underbrace{\sum_{i=1}^{n} (-1)^{i+j} \det\left(A \setminus \left\{\begin{matrix}\text{row } i, \\ \text{col } j\end{matrix}\right\}\right) b_i}_{= \det(A \text{ with col } j \text{ replaced by } \bar{b})} \qquad \blacksquare$$

Laplace expansion along column j

These formulas for A^{-1} and Cramer's rule for solving $A\bar{x} = \bar{b}$ are computationally useless unless n is very small, but theoretically very useful because they only involve polynomial functions of the entries of A together with dividing by $\det A$, a nonzero polynomial function of the entries of A. As an example ...

COR: If $\bar{a} \in \underset{\mathbb{R}^n}{\underset{\cap}{U}}{}^{\text{open}} \xrightarrow{\bar{f}} \mathbb{R}^n$ has f in $C^1(U)$ and $D\bar{f}(\bar{x})$ invertible on U, then the local inverse function $\underset{\mathbb{R}^n}{\underset{\cap}{V}}{}^{\text{open}} \xrightarrow{\bar{g} = \bar{f}^{-1}} U$ is also in C^1.

proof: Letting $\bar{b} = f(\bar{a})$, recall that $\bar{g} \circ \bar{f} = 1_U$

$$\overset{\text{Chain Rule}}{\Longrightarrow} [D\bar{g}(\bar{b})][D\bar{f}(\bar{a})] = I_n$$

i.e. $[D\bar{g}(\bar{b})] = [D\bar{f}(\bar{a})]^{-1}$

But same holds for all $\bar{x}$ in U and $\underset{\bar{x} = \bar{g}(\bar{y})}{\bar{y} = \bar{f}(\bar{x})} \in V$, i.e.
$$[D\bar{g}(\bar{y})] = [D\bar{f}(\bar{x})]^{-1} = [D\bar{f}(\bar{g}(\bar{y}))]^{-1}$$

(4)

We showed (in the proof of Inv. Function Thm) that $\bar{g}$ is continuous, hence since $\bar{f}$ being C^1 means all $\frac{\partial f_i}{\partial x_j}(\bar{x})$ are continuous, also the matrix entries of $[D\bar{f}(\bar{g}(\bar{y}))] = \left[\frac{\partial f_i}{\partial x_j}(\bar{g}(\bar{y}))\right]$ are continuous,

and thus $$[D\bar{g}(\bar{y})] = \frac{1}{\det\left[\frac{\partial f_i}{\partial x_j}(\bar{g}(\bar{y}))\right]} \operatorname{adj}\left[\frac{\partial f_i}{\partial x_j}(\bar{g}(\bar{y}))\right]^T$$ has all entries continuous,

i.e. $\bar{g}(\bar{y})$ is in $C^1(V)$ ∎

nonzero

Using sum, product, quotient and chain rules for differentiation, one can generalize this from C^1 to C^k as follows.

EXERCISE: (i) Given $f, g : U^{\text{open}} \to \mathbb{R}$, $U \subset \mathbb{R}^n$, with $f, g \in C^k(U)$, prove that $f + g \in C^k(U)$.

(ii) With same hypotheses, show $f \cdot g \in C^k(U)$ (HINT: Induct on k)

(iii) With the additional hypothesis that $g(\bar{x}) \neq 0 \ \forall \bar{x} \in U$, show $\frac{f}{g} \in C^k(U)$. (same hint)

(iv) Given $U^{\text{open}} \xrightarrow{\bar{f}} V^{\text{open}} \xrightarrow{g} \mathbb{R}$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, with $\bar{f} \in C^k(U)$, $g \in C^k(V)$, prove that $g \circ \bar{f} \in C^k(U)$ (same hint)

(v) Given inverse functions $U^{\text{open}} \underset{\bar{g} = \bar{f}^{-1}}{\overset{\bar{f}}{\rightleftarrows}} V^{\text{open}}$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^n$, with $\bar{f} \in C^k(U)$,

prove that $\bar{g} = \bar{f}^{-1} \in C^k(U)$ (same hint)