(70)

L-integrals have some good, expected properties like...

easy; see pp. 506–507

PROP 4.11.14: (Linearity) If both f, g are L-integrable, so is $af+bg$ for $a,b\in\mathbb{R}$ and $\int_{\mathbb{R}^n}(af+bg)\,|d^n\bar{x}|$

PROP 4.11.15: $\left.\begin{array}{l} f \text{ L-integrable} \\ g \text{ R-integrable}\end{array}\right\} \Rightarrow fg$ L-integrable

proof: Write $f \underset{a.e.}{=} \sum_k f_k$ with $\sum_k \int |f_k|\,|d^n\bar{x}|$ convergent

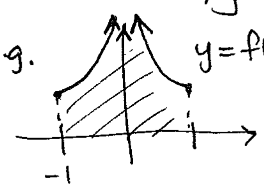
and let $M := \sup\{|g(\bar{x})| : \bar{x}\in\mathbb{R}^n\} < \infty$ since g must be bounded

Then $fg \underset{a.e.}{=} \sum_k \underbrace{f_k g}_{\text{each R-integrable}}$, with $\sum_k \int |f_k g|\,|d^n\bar{x}|$

$$\leq \underbrace{\sum_k M \int |f_k|\,|d^n\bar{x}|}_{\text{convergent!}}$$

∎

REMARK: It can fail for f, g L-integrable, not R-integrable

e.g. 

$y = f(x) = g(x) = \frac{1}{\sqrt{x}} \cdot 1_{[-1,1]}(x)$ is L-integrable,

but $f(x)g(x) = \frac{1}{x} \cdot 1_{[-1,1]}$ is not

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not hard; see pp. 507–508

PROP 4.11.16: f, g L-integrable with $f \underset{a.e.}{\leq} g \Rightarrow \int_{\mathbb{R}^n} f\,|d^n\bar{x}| \leq \int_{\mathbb{R}^n} g\,|d^n\bar{x}|$.

takes work; see App. A.22

THM 4.11.20 (Fubini) If $\underset{(\bar{x},\,\bar{y})}{\mathbb{R}^n\times\mathbb{R}^m} \xrightarrow{f} \mathbb{R}$ is L-integrable then $f_x(\bar{y}) = \int_{\mathbb{R}^n} f(\bar{x},\bar{y})\,|d^n\bar{x}|$

is defined for almost $\bar{y}\in\mathbb{R}^m$, and L-integrable, and

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f(\bar{x},\bar{y})\,|d^n\bar{x}|\,|d^m\bar{y}| = \int_{\mathbb{R}^m}\Big(\underbrace{\int_{\mathbb{R}^n} f(\bar{x},\bar{y})\,|d^n\bar{x}|}_{f_x(\bar{y})}\Big)\,|d^m\bar{y}|$$

THM 4.11.21 (Change-of-variable) With suitable hypotheses on a parametrization $X \xrightarrow{\Phi} Y$

one has for $Y \xrightarrow{f} \mathbb{R}$ L-integrable that $X \xrightarrow{f\circ\Phi} \mathbb{R}$ is L-integrable,

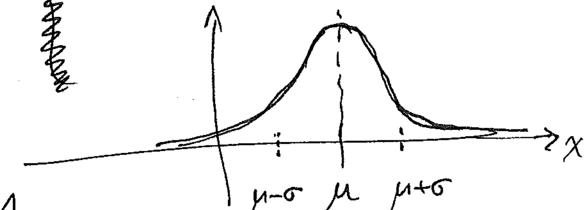
and $\int_X (f\circ\Phi)(\bar{x})\,|\det D\Phi(\bar{x})|\,|d^n\bar{x}| = \int_Y f(\bar{y})\,|d^n\bar{y}|$

(77) (EXAMPLE 4.11.22)

(4) A tricky one that appears in probability is to show that the Gaussian probability density $\mu(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ on $\mathbb{R}$

with mean μ

standard deviation σ



is properly normalized, i.e. $\int_{\mathbb{R}^1} \mu(x)\,|dx| \overset{?}{=} 1$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mathbb{R}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} |dx|$$

change-of-variable renaming $\frac{x-\mu}{\sqrt{2}\,\sigma}$ as x

Want $\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-x^2} |dx| \overset{?}{=} 1$

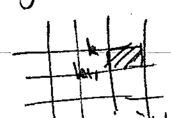
i.e. $\int_{\mathbb{R}} e^{-x^2} |dx| \overset{?}{=} \sqrt{\pi}$

problematic, since $e^{-x^2} \neq \frac{d}{dx} f(x)$ for some elementary function $f(x)$ (although this isn't obvious!)

Trick: $\left(\int_{\mathbb{R}} e^{-x^2} |dx|\right)^2 = \left(\int_{\mathbb{R}} e^{-x^2} |dx|\right)\left(\int_{\mathbb{R}} e^{-y^2} |dy|\right) = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} |dx||dy|$

using Fubini, once we know $e^{-(x^2+y^2)}$ is L-integrable:

$$e^{-(x^2+y^2)} = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \underbrace{1_{[j,j+1]\times[k,k+1]}(x,y) \cdot e^{-(x^2+y^2)}}_{f_{j,k}(x,y)}$$



Note $\sum_{j,k} \int \left|f_{j,k}(x,y)\right| |dx||dy| \leq \sum_{j,k} e^{-j^2} e^{-k^2} = \left(\sum_j e^{-j^2}\right)^2$

$e^{-j^2} = \frac{1}{e^{j^2}} \leq \frac{1}{e^j} \leq \frac{1}{2^j}$ $\qquad \leq 2 \underbrace{\sum_{j=0}^{\infty} \frac{1}{2^j}}_{\text{finite}}$

continue with polar coordinate change-of-variable (?)

$x = r\cos\theta$
$y = r\sin\theta$

$$\int_{\theta=0}^{\theta=2\pi} \left(\int_{r=0}^{r=\infty} e^{-r^2} \cdot r\,dr \right) d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2} \right]_{r=0}^{r=\infty} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} 1 \cdot d\theta = \frac{1}{2} \cdot 2\pi = \pi.$$

Hence $\left(\int_{\mathbb{R}} e^{-x^2} |dx|\right)^2 = \pi$

so $\int_{\mathbb{R}} e^{-x^2} |dx| = \sqrt{\pi}$ ✓

More good properties of L-integrals...

replaces f_k being R-integ with L-integ in defn of L-integ. Hard; see App. A.22

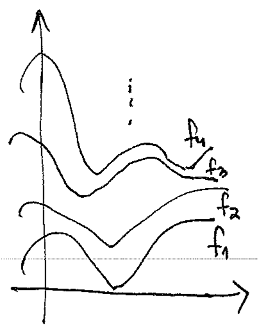
THM 4.11.17: If $\{f_k\}$ are L-integrable and $\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\bar{x})|\,|d^n\bar{x}|$ converges,

then $f(\bar{x}) = \sum_{k=1}^{\infty} f_k(\bar{x})$ converges a.e. and f is L-integrable

with $\int_{\mathbb{R}^n} f(\bar{x})\,|d^n\bar{x}| = \sum_{k=1}^{\infty} f_k(\bar{x})\,|d^n\bar{x}|$.

$\Rightarrow$ **THM 4.11.18** (Monotone convergence theorem)

If $0 \underset{a.e.}{\le} f_1 \underset{a.e.}{\le} f_2 \le \dots$ and $\{f_k\}$ are L-integrable



with $\sup_k \int_{\mathbb{R}^n} f_k(\bar{x})\,|d^n\bar{x}|$ finite, then

$f(\bar{x}) = \lim_{k\to\infty} f_k(\bar{x})$ ~~eeinverges~~ converges a.e. and f is L-integrable

with $\int_{\mathbb{R}^n} f(\bar{x})\,|d^n\bar{x}| = \sup_k \int_{\mathbb{R}^n} f_k(\bar{x})\,|d^n\bar{x}|$.

proof: Immediate from THM 4.11.17 taking $f = f_1 + (f_2 - f_1) + (f_3 - f_2) + \dots$ ∎

A very important, flexible result; see App. A.22 for proof

THM 4.11.19 (Dominated convergence theorem)

If $\{f_k\}$ are L-integrable and $|f_k(\bar{x})| \le F(x)$ $\forall k$ for some L-integrable F,

and if $f(\bar{x}) = \lim_{k\to\infty} f_k(\bar{x})$ converges a.e.,

then f is L-integrable and $\int_{\mathbb{R}^n} f(\bar{x})\,|d^n\bar{x}| = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(\bar{x})\,|d^n\bar{x}|$.

$\Downarrow$

THM 4.11.23 If • $\int_{\mathbb{R}^n} f(t,\bar{x})\,|d^n\bar{x}|$ exists $\forall t$

and • $\frac{\partial}{\partial t} f(t,\bar{x})$ exists for almost all $\bar{x}$

and • $\exists$ some L-integrable $\bar{g}(\bar{x})$ and $\epsilon > 0$ with

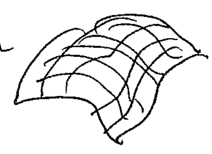
$0 < |s-t| < \epsilon \Rightarrow \left|\frac{f(s,\bar{x}) - f(t,\bar{x})}{s-t}\right| \le g(\bar{x})$

then $$\underbrace{\frac{\partial}{\partial t} \int_{\mathbb{R}^n} f(t,\bar{x})\,|d^n\bar{x}|}_{\text{and this exists!}} = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} f(t,\bar{x})\,|d^n\bar{x}|$$

Chapter 5 Arc length, surface area, etc.

How to make sense of $\underbrace{\mathrm{vol}_k(M)}_{k\text{-dimensional volume}}$ for a k-dimensional manifold $M \subset \mathbb{R}^n$ for $k<n$?

$k=1$: arc-length of a curve 

$k=2$: surface area 

$\vdots$

§5.1 The parallelogram (=parallelepiped) case

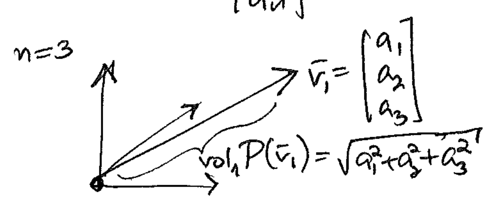
Given $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k \in \mathbb{R}^n$,
and the parallelepiped $P(\bar{v}_1, \dots, \bar{v}_k) := \left\{ \sum_{i=1}^{k} c_i v_i \ : \ c_i \in [0,1] \right\}$,
we know how to define its k-dimensional volume in some cases already,

e.g. if $k=n$, $P(\bar{v}_1, \dots, \bar{v}_n) = T(\underbrace{Q}_{\text{unit cube in } \mathbb{R}^n})$ where $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$, $\bar{x} \longmapsto A\bar{x}$

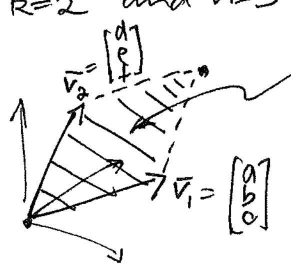
if $A = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}$

and $\mathrm{vol}_n P(\bar{v}_1, \dots, \bar{v}_n) = |\det A|$

if $k=1$, and $\bar{v}_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ then $\mathrm{vol}_1 P(\bar{v}_1) = |\bar{v}_1| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} = \sqrt{\bar{v}_1^T \bar{v}_1}$



if $k=2$ and $n=3$,



$\text{area} = \mathrm{vol}_2 P(\bar{v}_1, \bar{v}_2) = |\underbrace{\bar{v}_1 \times \bar{v}_2}_{= \det \begin{bmatrix} \bar{e}_1 & a & d \\ \bar{e}_2 & b & e \\ \bar{e}_3 & c & f \end{bmatrix}}| = \sqrt{(ae-bd)^2 + (af-cd)^2 + (bf-ce)^2}$

(8) The <u>right</u> way to generalize these uses the $\overset{n\times k}{\text{matrix}}$ $A = \begin{bmatrix} | & | & & | \\ \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_k \\ | & | & & | \end{bmatrix}$

and the associated Gram $\overset{\substack{\text{(symmetric)} \\ k\times k}}{\text{matrix}}$

$$A^TA = \underbrace{\begin{bmatrix} \text{—} & \bar{v}_1 & \text{—} \\ & \vdots & \\ \text{—} & \bar{v}_k & \text{—} \end{bmatrix}}_{k\times n} \underbrace{\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_k \\ | & & | \end{bmatrix}}_{n\times k} = \begin{bmatrix} \bar{v}_1^T\bar{v}_1 & \bar{v}_1^T\cdot\bar{v}_2 & \cdots & \bar{v}_1^T\bar{v}_k \\ \bar{v}_2^T\bar{v}_1 & \bar{v}_2^T\bar{v}_2 & & \vdots \\ \vdots & & \ddots & \\ \bar{v}_k^T\bar{v}_1 & \cdots & & \bar{v}_k^T\bar{v}_k \end{bmatrix}$$

<u>DEF'N 5.1.3</u>: $\quad \mathrm{vol}_k P(\bar{v}_1,\ldots,\bar{v}_k) := \sqrt{\det(A^TA)}$

<u>EXAMPLES</u>:

① When $k=n$ this agrees with the old formula $\mathrm{vol}_n P(\bar{v}_1,\ldots,\bar{v}_n) = |\det A|$,

since $\det(A^T) = \det A$ for square matrices A

THM 4.8.7 (<u>proof</u>: use row operations to write

$$A = E_1E_2\cdots E_t \underbrace{\left[\begin{array}{c|c} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\ \hline 0 & \begin{matrix}0 & & 0\\ & \ddots & \\ 0 & & 0\end{matrix} \end{array}\right]}_{\tilde{A}}$$

$\Rightarrow A^T = \tilde{A}^T E_t^T E_{t-1}^T \cdots E_2^T E_1^T$. Then use

$\det(XY) = \det X \cdot \det Y$, and check

$\det(E_i^T) = \det E_i \;\; \forall i$

$\det(\tilde{A}) = \det(\tilde{A}^T)$ ∎)

$$\Rightarrow \sqrt{\det(A^TA)} = \sqrt{\det(A^T)\det(A)}$$
$$= \sqrt{(\det(A))^2} = |\det(A)|.$$

② When $k=1$, it agrees with our old $\mathrm{vol}_1 P(\bar{v}_1) = \sqrt{\bar{v}_1^T\bar{v}_1} = \sqrt{a_1^2+\ldots+a_n^2}$

③ Checking it agrees when $k=2$, $n=3$ with our old $\mathrm{vol}_2 P(\bar{v}_1,\bar{v}_2)$
$= |\bar{v}_1\times\bar{v}_2|$

is EXER. 5.1.4.