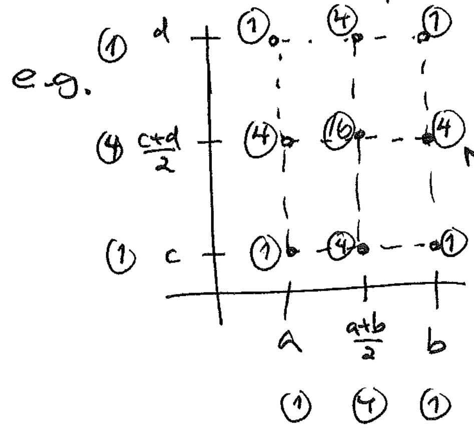
3/3/2017 →

(6D) In $\mathbb{R}^n$, there are product rules derived from the $\mathbb{R}^1$ rules,

e.g.



$$\int_a^b\int_c^d f(x,y)\,|dx\,dy| \approx \frac{(b-a)}{6}\frac{(c-d)}{6}\sum_{i=1}^{9} w_i f(x_i,y_i),$$

exact for f polynomial in x,y of

x-degree ≤ 3

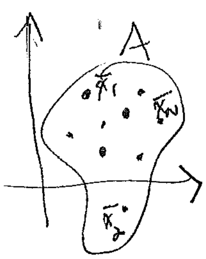
y-degree ≤ 3

(not hard to prove; see PROP 4.6.5 in book)

But one ~~lots~~ needs a lot of sample points as dimension n gets large (e.g. 3^n above)

A better approach is...

Monte Carlo method: To estimate $\int_A f(\bar{x})\,|d^n\bar{x}|$,



randomly pick N points $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N \in A$, and then we'd think

easier said than done!

$$\underset{\text{expected value of } f}{E(f)} = \frac{\int_A f(\bar{x})\,|d^n\bar{x}|}{\mathrm{vol}_n(A)} \approx \underbrace{\frac{1}{N}\left(f(\bar{x}_1)+\dots+f(\bar{x}_N)\right)}_{\text{call this } \bar{a}}$$

N^{th} sample mean

i.e. $\boxed{\int_A f(\bar{x})\,|d^n\bar{x}| \approx \mathrm{vol}_n(A)\cdot\bar{a}}$

How large to pick N? Try it first with some smallish N, to get

$$\bar{a} = \frac{1}{N}\sum_{i=1}^{N} f(\bar{x}_i)$$

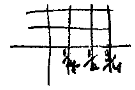
$$\bar{s}^2 = \frac{1}{N}\sum_{i=1}^{N}\left(f(\bar{x}_i)-\bar{a}\right)^2 = N^{\text{th}} \text{ sample variance} \approx \mathrm{var}(f)$$

i.e. $\bar{s} \approx \sigma(f)$

Then Central Limit Thm (THM 4.2.18) gives precise estimates on how ~~big~~ much bigger to choose N

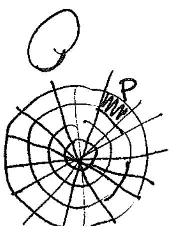
so that probability of $|\bar{a}-E(f)| \le \epsilon$ is small, in terms of $\sigma(f)$ $(\approx \bar{s})$ & N; see pp. 458-459 in book

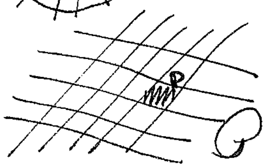
§4.7 Other pavings

(i.e. non-dyadic, not $\mathcal{D}_N(\mathbb{R}^n) = \{C_{\bar{k},N}\}$) 

Let's be more flexible now.

DEF'N 4.7.2: A paving of $X \subset \mathbb{R}^n$ is a collection $\mathcal{P} = \{P\}$ of
4.7.3 bounded (pavable?) subsets $P \subset X$ such that





(1) $X = \bigcup_{P \in \mathcal{P}} P$

(2) $\mathrm{vol}_n(P_1 \cap P_2) = 0$ if $P_1 \neq P_2$ in $\mathcal{P}$

(3) A bounded subset $Y \subset X$ has $\#\{P \in \mathcal{P} : P \cap Y \neq \emptyset\}$ finite

(4) $\mathrm{vol}_n(\partial P) = 0 \quad \forall P \in \mathcal{P}$

A sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of pavings of X is called a nested partition of X if

(1) $\mathcal{P}_{N+1}$ refines $\mathcal{P}_N$, i.e. every $P \in \mathcal{P}_{N+1}$ has $P \subset Q$ for some $Q \in \mathcal{P}_N$

(2) As $N \to \infty$, the P in $\mathcal{P}_N$ shrink to points,

i.e. $\displaystyle\lim_{N\to\infty} \sup_{P \in \mathcal{P}_N} \{\text{diameter}(P)\} = 0$

where $\text{diameter}(P) := \sup\{|\bar{x} - \bar{y}| : \bar{x}, \bar{y} \in P\}$.

If we have $X \xrightarrow{f} \mathbb{R}$ bounded in values, support and a nested partition $\mathcal{P}_N$ of X, can define $U_{\mathcal{P}_N}(f) := \sum_{P \in \mathcal{P}_N} M_P(f)\,\mathrm{vol}_n(P)$

$$L_{\mathcal{P}_N}(f) := \sum_{P \in \mathcal{P}_N} m_P(f)\,\mathrm{vol}_n(P)$$

THM 4.7.4: Given X bounded in $\mathbb{R}^n$ and $\mathcal{P}_N$ a nested partition of X,

a function $X \xrightarrow{f} \mathbb{R}$ has $f \cdot 1_X$ integrable (using our old dyadic Riemann sum definition)

$$\iff \lim_{N\to\infty} U_{\mathcal{P}_N}(f\cdot 1_X) = \lim_{N\to\infty} L_{\mathcal{P}_N}(f \cdot 1_X) = \int_X f(\bar{x})\,|d^n\bar{x}|$$

$$\overset{\text{DEF'N}}{=} \int_{\mathbb{R}^n} f \cdot 1_X\,|d^n\bar{x}|$$

((In other words, any nested partition works! to calculate $\int_X f |d^n\bar{x}|$)

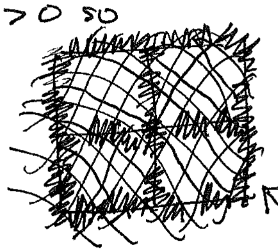
(62)

The proof (in Appendix A.18) is rather technical, and we'll skip it.

But the idea is to try and show, given $\epsilon > 0$ one can pick $N > 0$ so that $U_N(f) - L_N(f) < \frac{\epsilon}{2}$

and then pick $N'' > 0$ so that $|U_{\mathcal{P}_{N'}}(f) - U_N(f)| \leq \frac{\epsilon}{4}$

$|L_{\mathcal{P}_{N''}}(f) - L_N(f)| \leq \frac{\epsilon}{4}$



by ensuring most P in $\mathcal{P}_{N''}$ lie inside a single dyadic cube $C \in \mathcal{D}_N(\mathbb{R}^n)$ and those that straddle dyadic cubes have total volume small enough, relative to $\sup\{|f(\bar{x})| : \bar{x} \in X\}$.

3/6/2017

4.8, 4.9 Determinants & volume

We know $\det A$ for a square matrix tells us when A is invertible ($\det A \neq 0$).

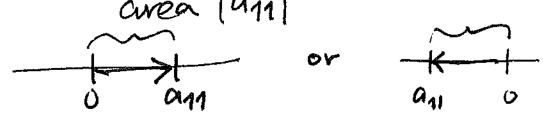
But what does it mean, as a number, when it's nonzero?

Recall for $n = 1, 2, 3$ we interpreted already (from Chap. 1)

$|\det A|$ = volume of parallelepiped spanned by columns $\bar{v}_1, \ldots, \bar{v}_n$ of $A = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}$

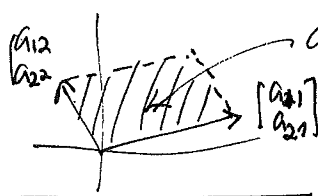
$n=1$

e.g. $A = [a_{11}]$



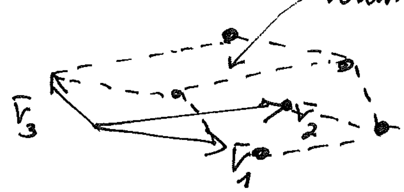
$n=2$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$



area $|a_{11}a_{22} - a_{21}a_{12}|$

$A = \begin{bmatrix} | & | & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\ | & | & | \end{bmatrix}$

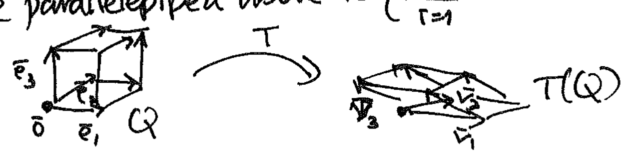


volume $= |(\bar{v}_1 \times \bar{v}_2) \cdot \bar{v}_3|$

$= \left|\det \begin{bmatrix} | & | & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\ | & | & | \end{bmatrix}\right|$

Now we'll prove it in general.

Note that if A is $n \times n$ matrix for $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ a linear transformation then the parallelepiped above is $\left\{\sum_{i=1}^n c_i \bar{v}_i : c_i \in [0,1]\right\} = T(Q)$ where $Q = \left\{\sum_{i=1}^n c_i \bar{e}_i : c_i \in [0,1]\right\}$



unit cube (of volume 1)