Math 5285 Honors fundamental structures of algebra
Fall 2018, Vic Reiner
Midterm exam 1- Due Wednesday October 10, in class

Instructions: There are 6 problems. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total, 10 points each part) Let A be a matrix in $\mathbb{R}^{m\times n}$.

(a) Show that A has a *right-inverse* R (that is, R in $\mathbb{R}^{n\times m}$ with $AR = I_m$) if and only if one can row-reduce A to a matrix in *row-echelon* form having no zero rows.

(b) Prove that A in $\mathbb{R}^{m\times n}$ never has a *unique* right-inverse if $m < n$.

2. (20 points total, 5 points each) Define a set of matrices

$$G_n := \{A \in \mathbb{C}^{n\times n} : A^T A = \pm I_n\}.$$

(a) Prove that G_n is a *subgroup* of $GL_n(\mathbb{C})$.

(b) Prove that every A in G_n has $\det A$ lying in

$$\begin{cases} \{+1,-1,+i,-i\} & \text{if } n \text{ is odd,} \\ \{+1,-1\} & \text{if } n \text{ is even.} \end{cases}$$

Also show, by examples that all six possibilities can occur. That is, for every odd n, exhibit examples of four matrices A in G_n with $\det A = +1, -1, +i, -i$, and for every even n, exhibit examples of two matrices A in G_n with $\det A = +1, -1$.

(c) Show $H_n := \{A \in G_n : \det A = +1\}$ is a *subgroup* of G_n, and a *normal* subgroup.

(d) Prove the index $[G_n : H_n]$ (= the number of different cosets gH_n in G_n) equals

$$\begin{cases} 4 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

3. (20 points total, 5 points each) Prove, or disprove via explicit counterexamples, the following assertions about the *orders* $\mathrm{ord}(g) := |\langle g\rangle| = |\{1, g, g^2, \ldots\}|$ of elements of *any* finite group G.

(a) For integers i, j, if i divides j, then $\mathrm{ord}(g^j)$ divides $\mathrm{ord}(g^i)$.

(b) For integers i, j, if i divides j, then $\mathrm{ord}(g^i)$ divides $\mathrm{ord}(g^j)$.

(c) For integers i, j, one has that $\mathrm{ord}(g^{i+j})$ divides $\mathrm{ord}(g^i) \cdot \mathrm{ord}(g^j)$.

(d) For g, h in G, one has that $\mathrm{ord}(gh)$ divides $\mathrm{ord}(g) \cdot \mathrm{ord}(h)$.

4. (20 points) Let A in $\mathbb{R}^{m\times n}$ have a row-reduced echelon form with r pivot ones. Prove that there exist matrices P in $GL_m(\mathbb{R})$ and Q in $GL_n(\mathbb{R})$ for which

$$PAQ = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where the matrix on the right contains r entries equal to one in its upper left.

5. (10 points total, 5 points each) Let $\phi : G_1 \longrightarrow G_2$ be a homomorphism between two groups G_1, G_2.

(a) Prove that for H_2 any subgroup of G_2, the subset

$$\phi^{-1}(H_2) := \{g \in G_1 : \phi(g) \in H_2\}$$

is a subgroup of G_1.

(b) Show H_2 a *normal* subgroup of G_2 implies $\phi^{-1}(H_2)$ a *normal* subgroup of G_1.

6. (10 points) Write down an isomorphism between the circle group

$$S^1 := \{z \in \mathbb{C}^\times : |z| = 1\}$$

and the group

$$SO_2 := \{A \in \mathbb{R}^{2\times 2} : A^T A = I_2, \det A = +1\}.$$

Make sure to explain why your map is an isomorphism.