Name: Vic Reiner - ANSWER KEY

Signature: ____________________

Math 5651 Lecture 003 (V. Reiner) Midterm Exam I
Thursday, February 25, 2016

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	10/10
2.	15/15
3.	15/15
4.	15/15
5.	15/15
6.	15/15
7.	15/15
Total:	100/100

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = B_1 \sqcup \cdots \sqcup B_n \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)$$

and Bayes Theorem $\mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$

$$\mathbf{E}X = \sum_k k \cdot f(k) \quad \text{for a discrete random variable with p.f. } f(k)$$

$$X = \mathrm{Bin}(n,p) \text{ has p.f. } f(k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k \in \{0,1,2,\ldots,n\}$$

$$X = \mathrm{Hypergeom}(A,B,n) \text{ has p.f. } f(k) = \binom{A}{k}\binom{B}{n-k} \Big/ \binom{A+B}{n} \text{ for } k \in \{0,1,2,\ldots,\min\{A,n\}\}$$

$$X = \mathrm{Poi}(\lambda) \text{ has p.f } f(k) = e^{-\lambda}\frac{\lambda^k}{k!} \text{ for } k \in \{0,1,2,\ldots\}$$

Problem 1. *(10 points)* When rolling a fair 6-sided dice having $1, 2, 3, 4, 5, 6$ on its sides n times, consider the two events A, B where A is rolling the same number every time, and B is rolling an odd number every time. Are A and B dependent or independent? You must support your answer with calculations to receive any credit.

($\frac{4}{10}$ for ignoring n and doing only $n=1$)

($\frac{8}{10}$ for picking some $n>1$)

(3pts)

$$Pr(A) = \sum_{i=1}^{6} Pr(\text{rolling } i \text{ every time}) = 6\cdot\left(\tfrac{1}{6}\right)^n = \frac{1}{6^{n-1}}$$

(3pts)

$$Pr(B) = \left(\tfrac{3}{6}\right)^n = \frac{1}{2^n}$$

$$Pr(A)\cdot Pr(B) = \frac{1}{2^n}\cdot\frac{1}{6^{n-1}}$$

$$Pr(A\cap B) = \sum_{i\in\{1,3,5\}} Pr(\text{rolling } i \text{ every time}) = 3\cdot\left(\tfrac{1}{6}\right)^n = \frac{3}{6^n}$$

So A,B are independent $\overset{(4pts)}{\iff} Pr(A\cap B) = Pr(A)Pr(B)$

$$\iff \frac{3}{6^n} = \frac{1}{2^n}\cdot\frac{1}{6^{n-1}}$$

$$\iff \frac{1}{2} = \frac{3}{6} = \frac{1}{2^n}$$

$$\iff n = 1$$

Problem 2. *(15 points)* If I choose a rearrangement of the 9 letters in the word "DISMISSED" into a possibly nonsensical string of 9 letters, with all rearrangements equally likely, then what is the probablity that *at least* one (and possibly more than one) of the following three events occurs?:

- The three letters "SSS" appear all adjacent. ← call this event S
- The two letters "II" appear adjacent. ← call this event I
- The two letters "DD" appear adjacent. ← call this event D

DISMISSED has
3 S's
2 I's
2 D's
1 M's
1 E's

(5pts)

Want $Pr(S\cup I\cup D) = Pr(S)+Pr(I)+Pr(D) - \big(Pr(S\cap I)+Pr(S\cap D)+Pr(I\cap D)\big) + Pr(S\cap I\cap D)$

$$= \frac{1}{\binom{9}{3\ 2\ 2\ 1\ 1}}\left[\underbrace{\binom{7}{1\ 2\ 2\ 1\ 1}}_{\text{"SSS" as a "superletter"}} + \underbrace{\binom{8}{3\ 1\ 2\ 1\ 1}}_{\text{"II" as a "superletter"}} + \underbrace{\binom{8}{3\ 2\ 1\ 1\ 1}}_{\text{"DD" as a "superletter"}} - \left(\binom{6}{1\ 1\ 2\ 1\ 1} + \binom{6}{1\ 2\ 1\ 1\ 1} + \binom{7}{3\ 1\ 1\ 1\ 1}\right) + \binom{5}{1\ 1\ 1\ 1\ 1}\right]$$

S's I's D's M's E's

(5pts)

"SSS" "II" "SSS" "DD" "II" "DD"

"SSS" "II" "DD"

(5pts)

8/15 for only calculating $Pr(S)$, $Pr(I)$, $Pr(D)$ separately, or saying $Pr(S\cup I\cup D) = Pr(S)+Pr(I)+Pr(D)$

Problem 3. *(15 points)* Your friend Bill Shakespeare tells you that he has written a new play, either a comedy or a tragedy. However, you know that whenever he says this, it is *always* written by someone else, a ghost-writer which is either Chris Marlowe, Frank Bacon, or Benny Jonson, each with their own fixed frequencies for writing tragedies versus comedies:

	Marlowe	Bacon	Jonson
fraction of time used as ghost-writer	1/2	1/4	1/4
% chance of writing a tragedy	90%	50%	20%

a. (5 points) Before he shows you the play, what is the probability that it is a comedy? call this event C

$$\Pr(C) = \Pr(C|\text{Marlowe})\Pr(\text{Marlowe}) + \Pr(C|\text{Bacon})\Pr(\text{Bacon}) + \Pr(C|\text{Jonson})\Pr(\text{Jonson})$$

$$= (0.10)\left(\tfrac{1}{2}\right) + (0.50)\left(\tfrac{1}{4}\right) + (0.80)\left(\tfrac{1}{4}\right)$$

$$\left(= 0.375 = \tfrac{3}{8}\right)$$

b. (10 points) You read it, and it is a tragedy. What is the probability that Bacon wrote it? call this event T

$$\Pr(\text{Bacon}|T) = \frac{\Pr(T|\text{Bacon})\Pr(\text{Bacon})}{\Pr(T)} = \frac{\Pr(T|\text{Bacon})\Pr(\text{Bacon})}{\Pr(T|\text{Bacon})\cdot\Pr(\text{Bacon}) + \Pr(T|\text{Marlowe})\cdot\Pr(\text{Marlowe}) + \Pr(T|\text{Jonson})\cdot\Pr(\text{Jonson})}$$

$$= \frac{(0.50)\left(\tfrac{1}{4}\right)}{(0.50)\left(\tfrac{1}{4}\right) + (0.90)\left(\tfrac{1}{2}\right) + (0.20)\left(\tfrac{1}{4}\right)}$$

$$\left(= \frac{0.125}{0.625} = \frac{1}{5}\right)$$

$\frac{8}{10}$ for switching "tragedy" to "comedy"

or use $\Pr(T) = 1 - \underbrace{\Pr(C)}_{=\text{answer from part (a)}}$

Problem 4. *(15 points total)* A group of 12 women and 10 men people pair off to make 11 pairs of swimming buddies. If all possible pairings equally likely, what is the probability that all the pairs are single-sex, that is, woman-woman or man-man?

call this event A

$S = \{\text{all possible pairings of 22 people}\}$

$$\Pr(A) = \frac{|A|}{|S|} = \frac{\overbrace{(11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1)}^{\#\text{ways to pair only women}}\overbrace{(9\cdot 7\cdot 5\cdot 3\cdot 1)}^{\#\text{ways to pair only men}}}{21\cdot 19\cdot 17\cdot 15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1} \quad \left(= \frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{21\cdot 19\cdot 17\cdot 15\cdot 13}\right)$$

(5pts)

$$\left(8/15 \text{ for } \frac{\binom{12}{6}\binom{10}{5}}{\binom{22}{11}} \text{ or } \frac{\binom{12}{222222}\binom{10}{22222}}{\binom{22}{22222222222}} \right)$$

Problem 5. *(15 points total)* Prove that if the events A, B, C are (jointly/mutually, not just pairwise) independent, then the events A^c and $B \cup C$ are also independent.

$$\Pr(A^c)\cdot\Pr(B\cup C) \overset{(3pts)}{=} (1-\Pr(A))\overbrace{(\Pr(B)+\Pr(C)-\Pr(B\cap C))}^{(3pts)}$$

B,C independent

$$\overset{(3pts)}{=} (1-\Pr(A))(\Pr(B)+\Pr(C)-\Pr(B)\Pr(C))$$

$$= \Pr(B)+\Pr(C)-\Pr(B)\Pr(C)-\Pr(A)\Pr(B)-\Pr(A)\Pr(C)+\Pr(A)\Pr(B)\Pr(C)$$

versus

$$\Pr(A^c\cap(B\cup C)) \overset{(3pts)}{=} \Pr((A^c\cap B)\cup(A^c\cap C))$$

$$= \Pr(A^c\cap B)+\Pr(A^c\cap C)-\Pr(\underbrace{(A^c\cap B)\cap(A^c\cap C)}_{=A^c\cap B\cap C})$$

$$= \Pr(B)-\Pr(A\cap B)+\Pr(C)-\Pr(A\cap C)-(\Pr(B\cap C)-\Pr(A\cap B\cap C))$$

A,B,C independent

$$= \Pr(B)-\Pr(A)\Pr(B)+\Pr(C)-\Pr(A)\Pr(C)-\Pr(B)\Pr(C)+\Pr(A)\Pr(B)\Pr(C)$$

= same as above

So $\Pr(A^c\cap(B\cup C)) \overset{(3pts)}{=} \Pr(A^c)\Pr(B\cup C)$

and hence A^c, $B\cup C$ are independent

Problem 6. *(15 points total)* You have two coins in your pocket, one *fair* coin having probability of heads 1/2, and one *unfair* coin in which the probability of heads is 3/4. You pull one of the two out of your pocket, with either coin equally probable, and start flipping it, generating a sequence of heads and tails.

a. (5 points) What is the probability that in a total of 10 flips you get at most 8 tails?

($\frac{3}{5}$ for using $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4} = \frac{3}{8}$ as tails probability)

$$\Pr(\le 8 \text{ tails}) \overset{(2pts)}{=} 1-\Pr(\ge 9 \text{ tails})$$

$$\overset{(1pt)}{=} 1-\left[\Pr(\ge 9 \text{ tails}\mid\text{fair})\Pr(\text{fair})+\Pr(\ge 9 \text{ tails}\mid\text{unfair})\Pr(\text{unfair})\right]$$

($\frac{4}{5}$ for changing "at most" to "exactly")

$$\overset{(2pts)}{=} 1-\left[\left(\binom{10}{9}\frac{1}{2^{10}}+\binom{10}{10}\frac{1}{2^{10}}\right)\left(\frac{1}{2}\right)+\left(\binom{10}{9}\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right)^{1}+\binom{10}{10}\left(\frac{1}{4}\right)^{10}\right)\right]$$

($\frac{1}{5}$ for recognizing Bin(p) as relevant)

b. (5 points) What is the probability that the first heads occurs exactly on the 6^{th} flip?

$$\Pr\left(\begin{array}{c}1^{st}\text{heads on}\\ 6^{th}\text{ flip}\end{array}\right) \overset{(2pts)}{=} \Pr((T,T,T,T,T,H)) = \Pr((T,T,T,T,T,H)\mid\text{fair})\Pr(\text{fair})+\Pr((T,T,T,T,T,H)\mid\text{unfair})\cdot\Pr(\text{unfair})$$

$$\overset{(3pts)}{=} \left(\frac{1}{2^6}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^1\cdot\left(\frac{1}{2}\right)$$

$$\left(=\frac{1}{2^7}+\frac{3^5}{2\cdot 4^6}\right)$$

c. (5 points) After you pull it out of your pocket, you do one test flip, and get heads. What is the probability that you pulled the fair coin out of your pocket?

$$\Pr(\text{fair}\mid H) \overset{(3pts)}{=} \frac{\Pr(H\mid\text{fair})\Pr(\text{fair})}{\Pr(H\mid\text{fair})\Pr(\text{fair})+\Pr(H\mid\text{unfair})\Pr(\text{unfair})}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)} \qquad \left(=\frac{\frac{1}{4}}{\frac{5}{8}}=\frac{2}{5}\right)$$

Problem 7. *(15 points)* Assume $X = Poi(\lambda)$ is a Poisson random variable with mean λ, and let $Y = X^2$, so that Y only takes on the values k^2 for $k = 0, 1, 2, \ldots$, and

$$\mathbf{Pr}(Y = k^2) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

What is the expected value $\mathbf{E}(Y)$ of Y?

(Hint: I think it helps to rewrite $k^2 = k(k-1) + k$.)

$$E(Y) = \sum_{k=0}^{\infty} k^2 \cdot Pr(Y = k^2)$$

$$= \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} k^2 \cdot \frac{\lambda^k}{k!}$$

(5 pts)

$$= e^{-\lambda} \left(\sum_{k=0}^{\infty} \underbrace{k(k-1)}_{\text{vanishes when } k=0,1} \cdot \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \underbrace{k}_{\text{vanishes when } k=0} \cdot \frac{\lambda^k}{k!} \right)$$

$$= e^{-\lambda} \left(\sum_{k=2}^{\infty} k(k-1) \cdot \frac{\lambda^k}{k!} + \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \right)$$

(5 pts)

$$= e^{-\lambda} \left(\lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right)$$

$$= e^{-\lambda} \left(\lambda^2 \cdot e^{\lambda} + \lambda \cdot e^{\lambda} \right)$$

(5 pts)

$$= \lambda^2 + \lambda$$

$\Big(\frac{1}{15}$ for "$E(Y) = \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \frac{\lambda^{k^2}}{(k^2)!}$" which is false.$\Big)$