Math 8201 Graduate abstract algebra- Fall 2019, Vic Reiner
Midterm exam 2- Due Wednesday November 20, in class

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult. Indicate any outside sources that you use. **Note:** In any problem with multiple parts, feel free to assume the assertions in any previous part when doing the next part, even if you didn't complete them.

1. (20 points total; 5 points each part) Prove or disprove:

(a) (5 points) For any subgroup H of a group G, the abelianization H^{ab} is isomorphic to a subgroup of G^{ab}.

(b) (5 points) A vector space V over a field can be isomorphic to one of its own proper subspaces $U \subsetneq V$.

(c) (5 points) A vector space V is finite-dimensional if and only if its dual V^* is also finite-dimensional.

(d) (5 points) Let V be a vector space over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for a *prime* p, and let $V^+ = (V, +)$ denotes its additive group structure. Then a subset $U \subset V$ is an $\mathbb{F}_p$-subspace if and only if $U^+ \subset V^+$ is a subgroup.

2. (25 points total; 5 points each part) Let $\varphi : V \to V$ be a linear operator on a vector space V over a field.

(a) Show that if $\mathrm{im}(\varphi^{m+1}) = \mathrm{im}(\varphi^m)$ for some integer $m \geq 1$, then $\mathrm{im}(\varphi^N) = \mathrm{im}(\varphi^m)$ for all $N \geq m$.

(b) Show that if $\ker(\varphi^{m+1}) = \ker(\varphi^m)$, for some integer $m \geq 1$, then $\ker(\varphi^N) = \ker(\varphi^m)$ for all $N \geq m$.

(c) Assuming that V is finite-dimensional, show that $\mathrm{im}(\varphi^{m+1}) = \mathrm{im}(\varphi^m)$ if and only if $\ker(\varphi^{m+1}) = \ker(\varphi^m)$.

(d) Give an example of $\varphi : V \to V$ with V infinite-dimensional where the conclusion of part (c) fails.

(e) Assume that V has finite dimension n, and $\varphi : V \to V$ satisfies $\varphi^m = 0$ for some integer $m \geq 1$, meaning $\varphi^m(v) = \mathbf{0}$ for all v in V. Show that $\varphi^n = 0$.

3. (20 points) Let G be a finite group,
- with $|G| = pqr$ for primes $p < q < r$,
- with q *not* dividing $r - 1$, and
- containing a normal subgroup $N \lhd G$ having $|N| = p$.

Prove that G is cyclic.

4. (20 points total; 5 points each part) Recall that a sequence of groups and homomorphisms

$$\cdots \xrightarrow{f_{i+2}} G_{i+1} \xrightarrow{f_{i+1}} G_i \xrightarrow{f_i} G_{i-1} \xrightarrow{f_{i-1}} \cdots$$

is called *exact* if $\ker(f_i) = \mathrm{im}(f_{i+1})$ for all i.

(a) (5 points) Given a short exact sequence of finite groups $1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1$, prove $|B| = |A||C|$.

(b) (5 points) Given an exact sequence of finite groups

$$1 \longrightarrow G_\ell \xrightarrow{f_\ell} G_{\ell-1} \xrightarrow{f_{\ell-1}} \cdots \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0 \longrightarrow 1$$

prove that these two sequences are also exact for each m:

$$1 \longrightarrow G_\ell \xrightarrow{f_\ell} G_{\ell-1} \xrightarrow{f_{\ell-1}} \cdots \xrightarrow{f_{m+1}} G_m \xrightarrow{f_m} \mathrm{im}(f_m) \to 1$$

$$1 \longrightarrow \ker(f_{m-1}) \longrightarrow G_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0 \longrightarrow 1$$

(Here the map $\ker(f_{m-1}) \longrightarrow G_{m-1}$ is simply the inclusion of the kernel as a subgroup.)

(c) (5 points) Prove that if one is given an exact sequence as in part (b), then

$$|G_1||G_3||G_5| \cdots = |G_0||G_2||G_4| \cdots$$

(d) (5 points) Given an exact sequence of finite-dimensional vector spaces $\{V_i\}$ over a field $\mathbb{F}$

$$0 \xrightarrow{f_{\ell+1}} V_\ell \xrightarrow{f_\ell} V_{\ell-1} \xrightarrow{f_{\ell-1}} \cdots \xrightarrow{f_3} V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \xrightarrow{f_0} 0,$$

prove that this alternating sum vanishes:

$$\dim_{\mathbb{F}} V_0 - \dim_{\mathbb{F}} V_1 + \dim_{\mathbb{F}} V_2 - \cdots + (-1)^\ell \dim_{\mathbb{F}} V_\ell = 0.$$

5. (15 points total; 5 points each part)

Let G_1 be a finite group, and $G_1 \xrightarrow{\pi} G_2$ a surjective group homomorphism. Also, fix a prime p.

(a) Prove that every p-Sylow subgroup P_1 in G_1 has image $\pi(P_1)$ which is a p-Sylow subgroup of G_2.

(b) Conversely, prove that for every p-Sylow subgroup P_2 of G_2, there exists at least one p-Sylow subgroup P_1 of G_1 having $\pi(P_1) = P_2$.

(c) Prove the numbers $n_p(G_1), n_p(G_2)$ of p-Sylow subgroups in G_1, G_2, respectively, have $n_p(G_1) \geq n_p(G_2)$.