Math 8202 Graduate abstract algebra- Spring 2020, Vic Reiner
Midterm exam 2- Due Wednesday April 8, sent in PDF via email

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (30 points total; 5 points each)
True or false; prove or disprove.

(a) Let $\mathbb{K}/\mathbb{F}$ be a field extension. If $\mathbb{K}$ is the splitting field over $\mathbb{F}$ for some finite list of separable polynomials $\{f_1, \ldots, f_n\}$ in $\mathbb{F}[x]$, then $\mathbb{K}$ is the splitting field over $\mathbb{F}$ for a single separable polynomial $f(x)$ in $\mathbb{F}[x]$.

(b) Let $\mathbb{K}/\mathbb{F}$ be a field extension of *finite degree.* If $\mathbb{K}$ is the splitting field over $\mathbb{F}$ for some possibly infinite family $\{f_i\}_{i \in I}$ of separable polynomials in $\mathbb{F}[x]$, then $\mathbb{K}$ is the splitting field over $\mathbb{F}$ for a single separable polynomial $f(x)$ in $\mathbb{F}[x]$.

(c) Let $\mathbb{K}/\mathbb{F}$ be a field extensions in which $\mathbb{K}$ is the splitting field over $\mathbb{F}$ for an *inseparable* polynomial $f(x)$. Then $\mathbb{K}/\mathbb{F}$ cannot be a Galois extension.

(d) For $n = 10^9$, the polynomial $x^n - 1$ has exactly 100 irreducible factors when considered in $\mathbb{R}[x]$.

(e) For $n = 10^9$, the polynomial $x^n - 1$ has exactly 100 irreducible factors when considered in $\mathbb{Q}[x]$.

(f) If α lies in some extension over the field $\mathbb{F}$ with $[\mathbb{F}(\alpha) : \mathbb{F}] = n$ and k is a positive integer with $\gcd(k, n) = 1$, then $\mathbb{F}(\alpha^k) = \mathbb{F}(\alpha)$.

2. (10 points) Show that if $f(x)$ is irreducible of degree k in $\mathbb{F}[x]$ and $[\mathbb{K} : \mathbb{F}] = n$ has $\gcd(k, n) = 1$, then $f(x)$ remains irreducible when considered in $\mathbb{K}[x]$.

3. (30 points) Let $\zeta = \zeta_{100} := e^{\frac{2\pi i}{100}}$, and $\mathbb{K} = \mathbb{Q}(\zeta)$.

(a) (5 points) Is $\mathbb{K}/\mathbb{Q}$ Galois? Explain.

(b) (5 points) Compute $[\mathbb{K} : \mathbb{Q}]$.

(c) (5 points) Identify the isomorphism type of $\mathrm{Aut}(\mathbb{K}/\mathbb{Q})$ as a finite abelian group, that is, to which product of cyclic groups is it isomorphic?

(d) (5 points) How many intermediate subfields $\mathbb{L}$ are there with $\mathbb{Q} \subseteq \mathbb{L} \subseteq \mathbb{K}$, including $\mathbb{L} = \mathbb{Q}, \mathbb{K}$?

(e) (5 points) How many of the subfields $\mathbb{L}$ counted in part (d) have $\mathbb{L}/\mathbb{Q}$ Galois?

(f) (5 points) How many of the subfields counted in part (d) contain $\zeta + \zeta^{-1}$?

4. (10 points total) Let $f(x), g(x)$ lie in $\mathbb{F}[x]$ with $f(x)$ irreducible of degree n. Show that every irreducible factor in $\mathbb{F}[x]$ of the composite polynomial $f(g(x))$ will have degree divisible by n.

5. (20 points total; 5 points each) Let $\mathbb{K} = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$ for p, q, r three distinct prime numbers. It can be shown with a bit of tedious checking that $[\mathbb{K} : \mathbb{Q}] = 8$, but let's **assume that**.

(a) Identify the isomorphism type of the group $\mathrm{Aut}(\mathbb{K}/\mathbb{Q})$.

(b) How many intermediate subfields $\mathbb{L}$ are there with $\mathbb{Q} \subseteq \mathbb{L} \subseteq \mathbb{K}$, including $\mathbb{L} = \mathbb{Q}, \mathbb{K}$?

(c) How many of the subfields $\mathbb{L}$ counted in part (b) have $\mathbb{L}/\mathbb{Q}$ Galois?

(d) How many of the subfields counted in part (b) contain $\alpha := 10\sqrt{p} - 3/7\sqrt{r}$?