Math 8680 Fall 2022

Combinatorics of reflection groups and invariant theory

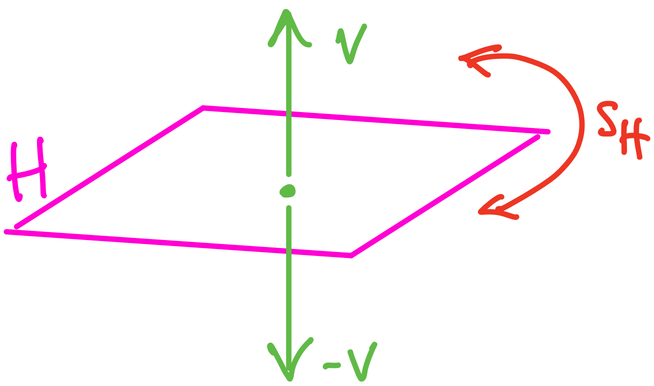
Syllabus items
- Office Hour Times ??
- Discord server, link on syllabus
- Homework: 5 problems by Dec. 1.
- Prerequisites:
  algebra, (a bit of) rep theory

Overview
(see also B&B §1.1, 1.2)

Let's start with ...

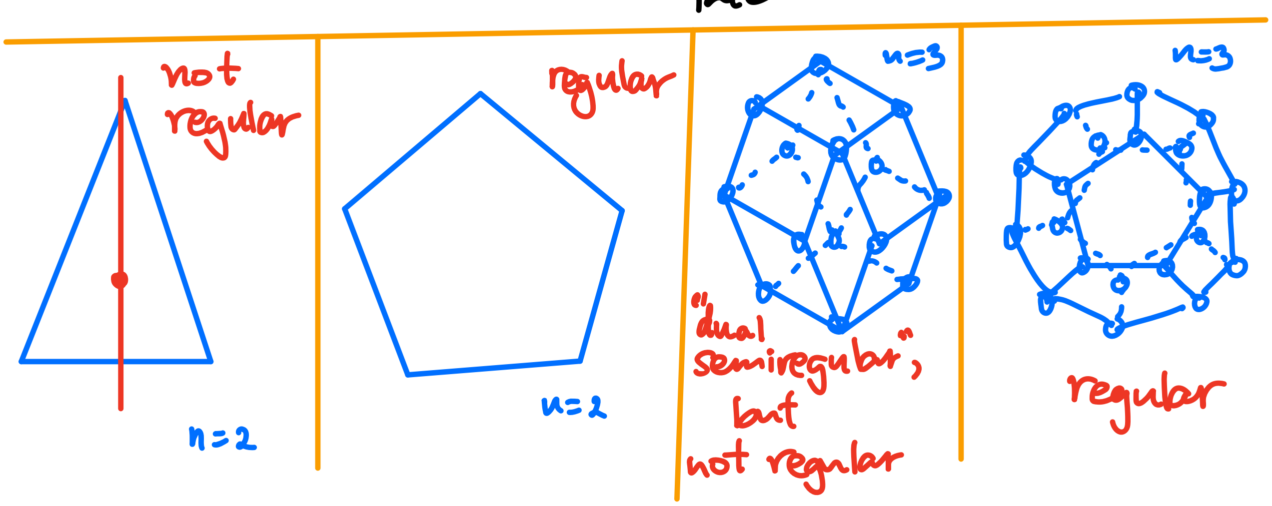
DEF'N: A real reflection group is a finite subgroup $W \subset GL_n(\mathbb{R}) = GL(V)$ with $V = \mathbb{R}^n$

generated by real (Euclidean) reflections
$s_H :=$ perpendicular reflection through some hyperplane H
(= codim 1 linear subspace)



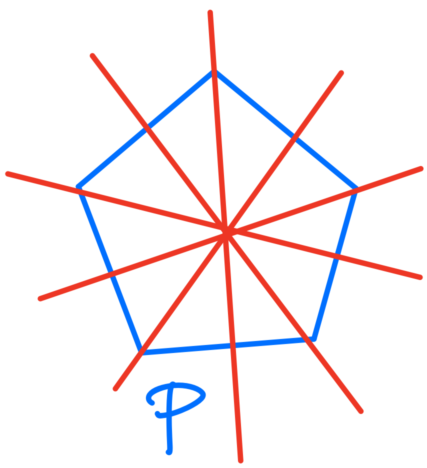
Sources of real reflection groups

- Lie theory & root systems of semisimple Lie algebras/groups
- Regular polytopes P := convex polytopes $P \subset \mathbb{R}^n = V$ whose linear symmetry group $W \subset GL(V)$ is transitive on all maximal flags of faces $\underbrace{F_0}_{\text{vertex}} \subset \underbrace{F_1}_{\text{edge}} \subset \underbrace{F_2}_{\text{polygon face}} \subset \dots \subset \underbrace{F_{n-1}}_{\text{facet}} \subset P$

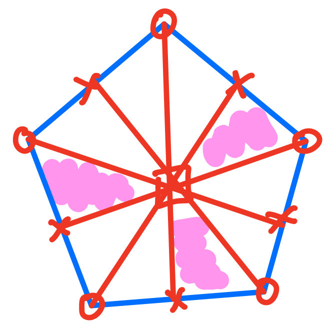


FACTS (see EXERCISE 9 in Portugal Summer School list)

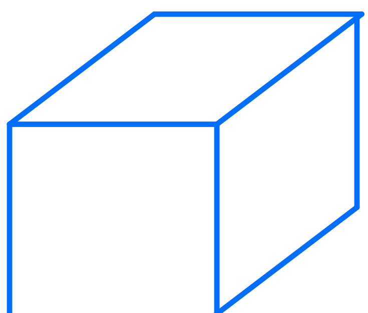
- W is a real reflection group, i.e. generated by s_H
- P is dissected by all the reflecting hyperplanes H into its barycentric subdivision SdP
- W acts simply transitively on maximal flags



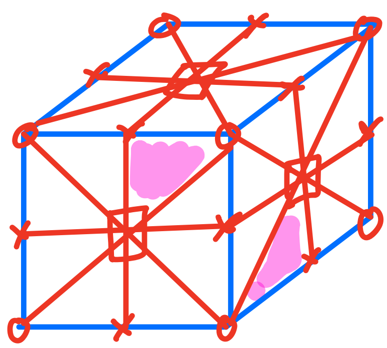
P



SdP

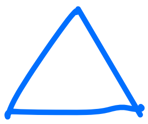


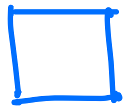
P



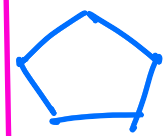
SdP

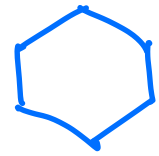
Regular polytope CLASSIFICATION:





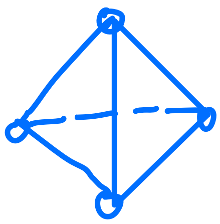
$n=2$

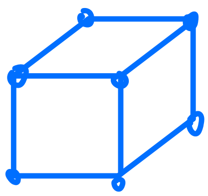


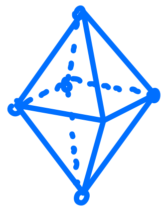


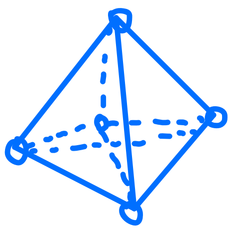
. . .

regular m-gons









$\vdots$

regular simplices $=$ convex hull of $e_1, \dots, e_n$ in $\mathbb{R}^n = V$ (translated by $-\frac{1}{n}(e_1 + \dots + e_n)$)

$\vdots$

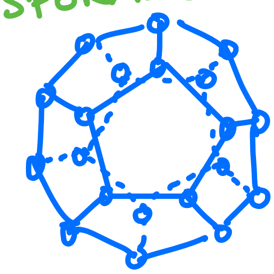
n-cubes $=$ convex hull of $\pm e_1 \pm e_2 \pm \dots \pm e_n$

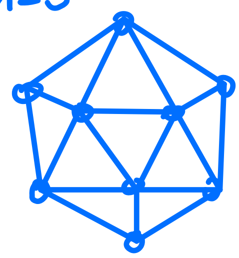
$\vdots$

n-cross polytopes $=$ convex hull of $\pm e_1$, $\pm e_2$, $\vdots$, $\pm e_n$

SPORADIC:

$n=3$





$n=4$:

Schläfli's regular 4-polytopes

- 600-cell
- 120-cell

- 24-cell

Two surprising and useful features for
W a real reflection group in $GL(V)$, $V=\mathbb{R}^n$

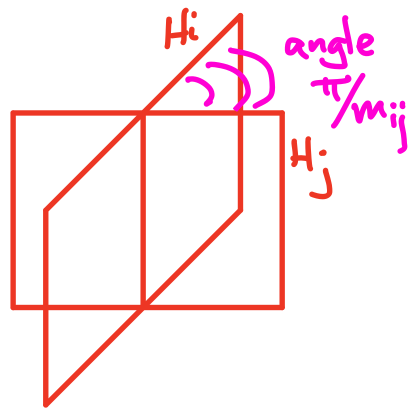
(1) Coxeter presentation:

Pick any chamber C_0
:= connected component of complement of all reflection hyperplanes H

Then the simple reflections $\{s_1, s_2, \ldots, s_n\} =: S$ through the walls $H_1, H_2, \ldots, H_n$ of C_0 give this presentation for W:

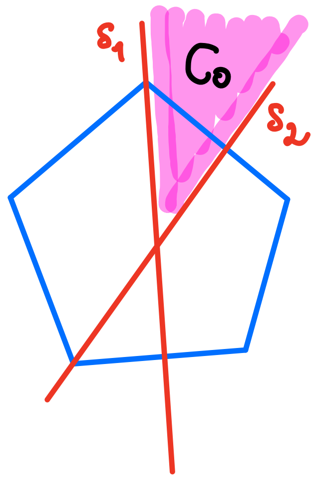
$$W \cong \langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$$

if H_i, H_j have dihedral angle $\frac{\pi}{m_{ij}}$

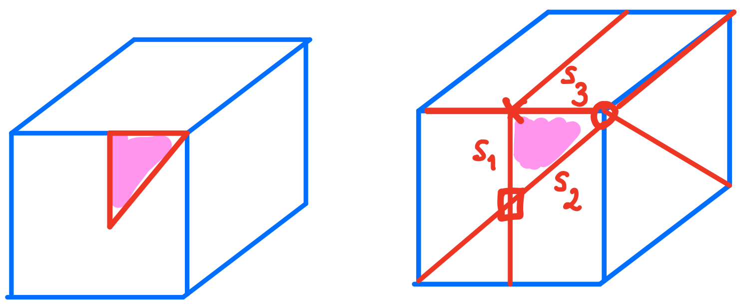


encoded via Coxeter diagram:

- nodes s_i
- edges $s_i \overset{m_{ij}}{—} s_j$
- omit edge when $m_{ij}=2$

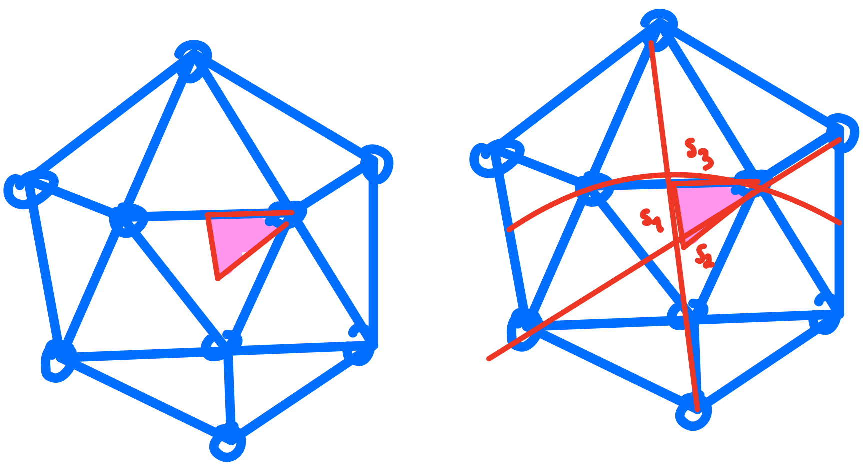


$$s_1 \overset{5}{—} s_2$$



$$s_1 \overset{4}{—} s_2 \overset{3}{—} s_3$$

$$(m_{13}=2)$$



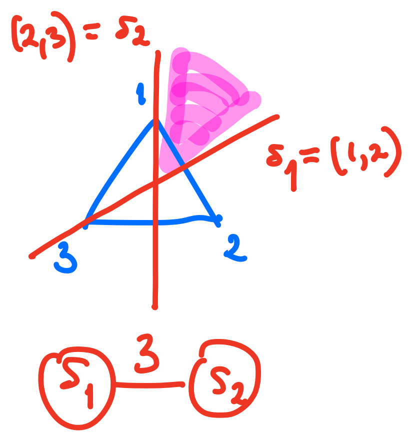
$$s_1 \overset{5}{—} s_2 \overset{3}{—} s_3$$

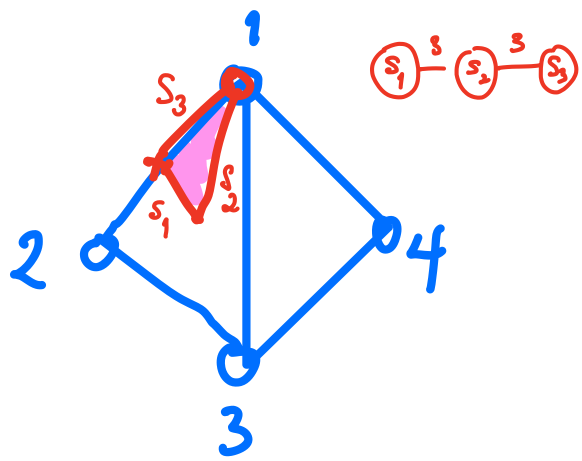
$$(m_{13}=2)$$

For regular simplices,

$W = \mathfrak{S}_n =$ symmetric group on $\{1, 2, \ldots, n\}$

$S = \{s_1, s_2, \ldots, s_{n-1}\}$ where $s_1 = (1,2)$, $s_2 = (2,3)$, …, $s_{n-1} = (n-1,n)$ simple transpositions





Implications of the Coxeter presentation

$$\mathfrak{S}_n = \left\langle s_1, s_2, \ldots, s_{n-1} \;\middle|\; \begin{array}{l} s_i^2 = 1 = (s_i s_j)^2 \text{ if } |i-j| \geq 2 \\ = (s_i s_{i+1})^3 \end{array} \right\rangle$$

generalize a lot of symmetric group combinatorics, particularly of inversions of permutations,

e.g. $$\sum_{w \in \mathfrak{S}_n} q^{\#\text{inversions of } w} = [n]!_q = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$
$$= [1]_q [2]_q [3]_q \cdots [n]_q$$

pairs $i<j$ with $w(i) > w(j)$

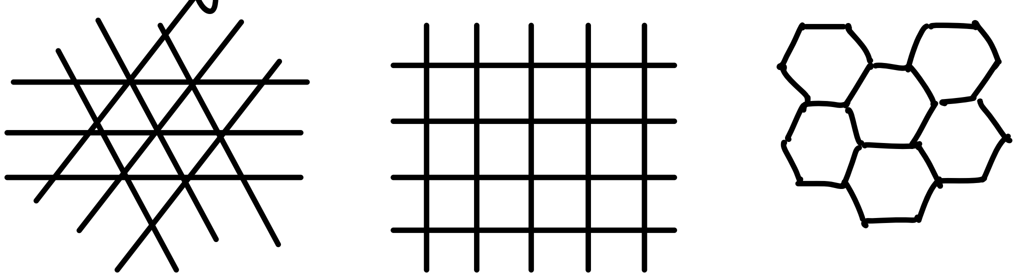
This suggests considering Coxeter groups more generally, i.e.

$$W \cong \langle \underset{\{s_1, \ldots, s_n\}}{\overset{\shortparallel}{S}} \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$$

with $m_{ij} \in \{2,3,4,\ldots\} \cup \{\infty\}$

It turns out to be natural, capturing

- affine symmetries of regular tessellations



- more Lie theory (Kac-Moody Lie algebras)

THEOREM (Coxeter 1934)

$\{$Finite Coxeter groups $W\}$ = $\{$real reflection groups$\}$

(those with a Coxeter presentation)

2^{nd} surprising feature of real reflection groups $W \subset GL(V)$, $V = \mathbb{R}^n$:

(2) Good invariant theory

Let $W \subset GL(V)$, $V = \mathbb{R}^n$ act on V^* and its basis $x_1, \ldots, x_n$ thought of as variables in $\mathbb{R}[x_1, \ldots, x_n] =: \mathbb{R}[\underline{x}]$.

Invariant theory asks:

(a) What does the W-invariant subalgebra

$$\mathbb{R}[\underline{x}]^W := \{ f(\underline{x}) \in \mathbb{R}[\underline{x}] : w(f(\underline{x})) = f(w\underline{x}) \ \forall w \in W \}$$

look like as a ring? Generators, relations?

(b) What does the rest of $\mathbb{R}[\underline{x}]$ look like as an $\mathbb{R}[\underline{x}]^W$-module? Generators, relations?

Structure of χ-isotypic components $\mathbb{R}[\underline{x}]^{W,\chi}$ as $\mathbb{R}[\underline{x}]^W$-module?

The answers are as simple as possible for reflection groups:

THEOREM (Shephard-Todd, 1955, Chevalley 1955)
For a real reflection group W,

(a) $\mathbb{R}[\underline{x}]^W = \mathbb{R}[f_1, f_2, \ldots, f_n]$ for some homogeneous algebraically independent $f_1, f_2, \ldots, f_n$
(i.e. n generators, no relations!)

(b) $\mathbb{R}[\underline{x}]$ is a free $\mathbb{R}[\underline{x}]^W$-module,
(no relations again)
with free basis elements given by any lifts of ...

(c) $\underbrace{\mathbb{R}[\underline{x}]/(f_1, f_2, \ldots, f_n)}_{\text{the coinvariant algebra}} \overset{\text{as } W\text{-representations}}{\cong} \mathbb{R}[W]$
= the regular representation of W

EXAMPLE

$W = \mathfrak{S}_3 \subset GL(V)$, $V = \mathbb{R}^3$

$V^* = \mathbb{R}^3$ with basis x_1, x_2, x_3

so $\mathfrak{S}_3$ permutes variables in $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, x_2, x_3]$

and $\mathbb{R}[\underline{x}]^W = \mathbb{R}[x_1, x_2, x_3]^{\mathfrak{S}_3}$ = symmetric polynomials

$$= \mathbb{R}[f_1, f_2, f_3]$$

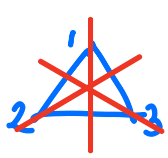
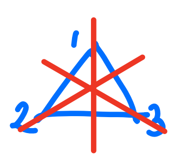
$f_1 = e_1 = x_1 + x_2 + x_3$, $f_2 = e_2 = x_1x_2 + x_1x_3 + x_2x_3$, $f_3 = e_3 = x_1x_2x_3$

elementary symmetric polynomials

Coinvariant algebra

$$\mathbb{R}[\underline{x}]/(f_1, f_2, f_3) = \mathbb{R}[x_1, x_2, x_3]/(x_1+x_2+x_3,\ x_1x_2+x_1x_3+x_2x_3,\ x_1x_2x_3)$$

$$= \operatorname{span}_{\mathbb{R}}\{\bar{1},\ |\ \bar{x}_1, \bar{x}_2,\ |\ \bar{x}_1^2, \bar{x}_2^2,\ |\ \overline{(x_1-x_2)(x_2-x_3)(x_1-x_3)}\}$$

degree	0	1	2	3
$\mathfrak{S}_3$-irreducible decomposition	$\chi^{(3)} = \mathbb{1}_{\mathfrak{S}_3}$	$\chi^{(2,1)}$ = 	$\chi^{(2,1)}$ = 	$\chi^{(1,1,1)} = \mathrm{sgn}_{\mathfrak{S}_3}$

The basic/fundamental degrees of W

$d_1 = \deg(f_1)$, $d_2 = \deg(f_2)$, ..., $d_n = \deg(f_n)$

predict shocking amounts of W's numerology

EXAMPLE Let $\ell_S(w) :=$ Coxeter group length of w in W

$= \min\{\ell : w = s_{i_1} s_{i_2} \cdots s_{i_\ell}, \ s_i \in S\}$

THEOREM:

$$\sum_{w \in W} q^{\ell_S(w)} = [d_1]_q [d_2]_q \cdots [d_n]_q$$

$W = \mathfrak{S}_n$

$$\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = [1]_q [2]_q \cdots [n]_q = [n]!_q$$

$n=3$:

w	$\mathrm{inv}(w)$
123	0
132	1
213	1
231	2
312	2
321	3

$1 + 2q + 2q^2 + q^3$

$= (1)(1+q)(1+q+q^2)$

$= [3]!_q$

EXAMPLE Let $V^w = \{v \in V = \mathbb{R}^n : w(v) = v\}$ for $w \in W$

an $\mathbb{R}$-linear subspace of V

THEOREM:

$$\sum_{w \in W} q^{\dim(V^w)} = (t+d_1-1)(t+d_2-1)\cdots(t+d_n-1)$$

$$\Big\downarrow \; W = \mathfrak{S}_n$$

$$\sum_{w \in \mathfrak{S}_n} q^{\#\text{cycles}(w)} = t(t+1)(t+2)\cdots(t+n-1)$$

$n=3$:

w	$\#\text{cycles}(w)$
$123 = (1)(2)(3)$	3
$132 = (1)(23)$	2
$213 = (12)(3)$	2
$231 = (123)$	1
$312 = (132)$	1
$321 = (13)(2)$	2

$t^3 + 3t^2 + 2t$

$= t(t+1)(t+2)$

The proof uses a THEOREM of Solomon on

W-invariants of $\mathbb{R}[x_1, \ldots, x_n] \otimes \underbrace{\Lambda\{x_1, \ldots, x_n\}}_{\text{exterior algebra on } x_1, x_2, \ldots, x_n}$

The Chevalley/Shephard-Todd result is really about complex reflection groups $W \subset GL_n(\mathbb{C}) = GL(V)$ for $V = \mathbb{C}^n$

which are finite subgroups of $GL(V)$ generated by complex reflections s

s fixes a hyperplane $H \subset \mathbb{C}$, and scales the line $H^\perp$ by some root-of-unity $\zeta \neq 1$ in $\mathbb{C}^\times$, i.e. s diagonalizes to $\begin{bmatrix} \zeta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$

THEOREM

For finite subgroups $W \subset GL(V)$, $V = \mathbb{C}^n$

T.F.A.E. (a) $\mathbb{C}[x_1, \ldots, x_n]^W = \mathbb{C}[f_1, \ldots, f_n]$ is polynomia

(b) $\mathbb{C}[\underline{x}]$ is a free $\mathbb{C}[\underline{x}]^W$-module

(c) W is a complex reflection group

Coxeter-Catalan combinatorics

deals with the many sets associated to an (irreducible) real reflection group W that have cardinality given by the

W-Catalan number $\quad Cat(W) := \dfrac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1 \cdot d_2 \cdot \cdots \cdot d_n}$

where $h := \max\{d_1, d_2, \ldots, d_n\}$

= the Coxeter number

= the multiplicative order of any Coxeter element $c := s_1 s_2 \cdots s_n$

$W = \mathfrak{S}_n$ acting irreducibly

$d_1 = 2,\ d_2 = 3,\ \ldots,\ d_{n-1} = n =: h$

$$Cat(\mathfrak{S}_n) = \frac{(n+2)(n+3)\cdots(2n)}{2 \cdot 3 \cdot \cdots \cdot n} = \frac{1}{n+1}\binom{2n}{n}$$

Catalan number

$GL_n(\mathbb{F}_q)$-analogues

One can also work over $\mathbb{F}_q$ and view $W = GL_n(\mathbb{F}_q) = GL(V)$ as a finite reflection group for $V = \mathbb{F}_q^n$

q-analogous to $\mathfrak{S}_n$.

THEOREM (L. E. Dickson 1911)

$$\mathbb{F}_q[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_q)} = \mathbb{F}_q[f_1, f_2, \ldots, f_n]$$ is a polynomial algebra,

where $f_1, f_2, \ldots, f_n$ are the coefficients of

$$\prod_{\substack{\text{all linear forms}\\ c_1x_1+\ldots+c_nx_n \text{ in } (\mathbb{F}_q^n)^*}} (t + c_1x_1 + \ldots + c_nx_n) = t^{q^n} + t^{q^{n-1}} f_1(x) + t^{q^{n-2}} f_2(x) + \ldots + t^{q^0} f_n(x)$$

Compare Dickson's Theorem

- $\mathbb{F}_q[x_1,\ldots,x_n]^{GL_n(\mathbb{F}_q)} = \mathbb{F}_q[f_1, f_2, \ldots, f_n]$

where $\prod(t + c_1x_1 + \ldots + c_nx_n) = t^{q^n} + t^{q^{n-1}} f_1(\underline{x}) + t^{q^{n-2}} f_2(\underline{x}) + \ldots + t^{q^0} f_n(\underline{x})$

all linear forms

$c_1x_1+\ldots+c_nx_n$ in $(\mathbb{F}_q^n)^*$

AND

- $\mathbb{R}[x_1,\ldots,x_n]^{\mathfrak{S}_n} = \mathbb{R}[e_1, e_2, \ldots, e_n]$

where

$$\prod_{i=1}^{n}(t+x_i) = t^n + t^{n-1} e_1(\underline{x}) + t^{n-2} e_2(\underline{x}) + \ldots + t^0 e_n(\underline{x})$$

There appears to be a lot of reflection group combinatorics happening for $GL_n(\mathbb{F}_q)$, including strange analogies with Coxeter-Catalan combinatorics?