Irreducibility, nondegeneracy, chamber geometry and finiteness
(Humphreys § 6.1, 6.3, 5.13, 6.2, 6.4)

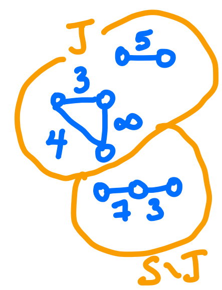
To understand more about (W,S) and when W is finite, let's understand some apparent ways in which the geom. rep'n

$$W \xrightarrow{\sigma} GL(V)$$

would fail to be irreducible, i.e. $\exists$ W-stable subspaces $\{0\} \subsetneq W' \subsetneq W$.

PROPOSITION: If the Cox. diagram for (W,S) is disconnected, say

$$S = J \sqcup S\setminus J \quad \text{with } m_{ij} = 2 \;\; \forall \; \begin{matrix} s_j \in J \\ s_i \in S\setminus J \end{matrix}$$



then $W \cong W_J \times W_{S/J}$ and the geom. rep'n is a direct sum:

$$\begin{tikzcd}
W \arrow[r, "\sigma"] \arrow[d, phantom, "\wr\wr"] & V \arrow[d, phantom, "\parallel"] \\
W_J \times W_{S/J} \arrow[r, "\sigma_J \oplus \sigma_{S/J}"] & V_J \oplus V_{S/J}
\end{tikzcd}$$

proof: One has W_J and $W_{S/J}$ centralizing each other, since their generators commute $s_i s_j = s_j s_i$.

Hence $W_J, W_{S/J} \lhd W$ and $W = W_J \cdot W_{S/J}$ $\Big\}$ since $S = J \cup S/J$.

Need to check $W_J \cap W_{S/J} = \{1\}$:

if $w \in W_J \cap W_{S/J}$ and $w \neq 1$,
pick $s_j \in J$ with $\ell(ws_j) < \ell(w)$, so $s_j \in T_R(w)$.
But picking a reduced word for w in $W_{S/J}$
shows $T_R(w) \subset W_{S/J}$, so $s_j \in J \cap W_{S/J}$ ↯

In the geom. rep'n $W \xrightarrow{\sigma} GL(V)$
one has $V_J := \mathrm{span}_{\mathbb{R}} \{\alpha_j\}_{s_j \in J}$ pointwise fixed by $W_{S/J}$
$V_{S/J} := \mathrm{span}_{\mathbb{R}} \{\alpha_i\}_{s_i \in S \setminus J}$ —— " —— by W_J

since $\sigma_i(\alpha_j) = \alpha_j - 2\underbrace{B(\alpha_j, \alpha_i)}_{=0}\alpha_i$ if $m_{ij} = 2$

$= \alpha_j$

Hence $V = V_J \oplus V_{S/J}$ is the direct sum of the geom rep'ns $W_J \to GL(V_J)$, $W_{S/J} \to GL(V_{S/J})$ ∎

DEF'N: Call (W,S) an irreducible Cox. system if the Cox. diagram is connected; reducible otherwise.

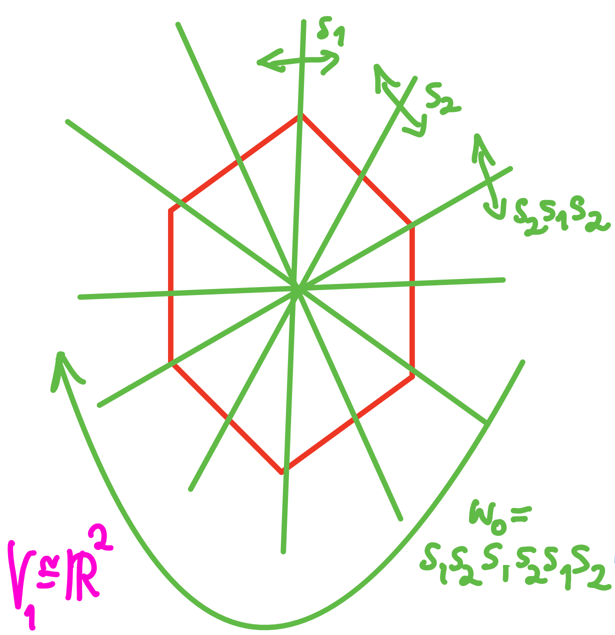
EXAMPLE (Björner–Brenti EXERCISE 1.1.2)

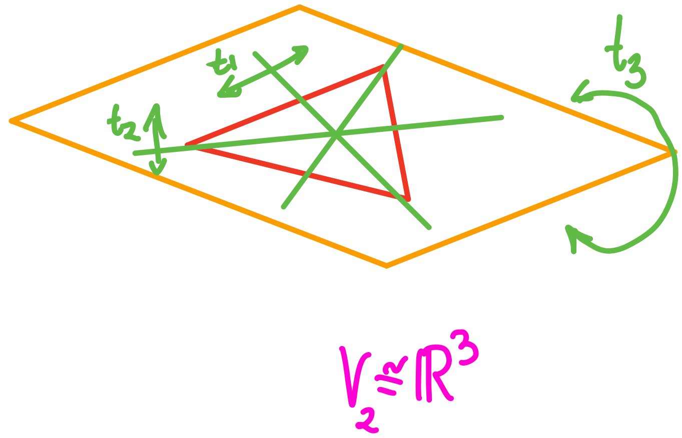
As an abstract group, $W =$ dihedral group of order 12 has both (!):

$$I_2(6) \cong W \cong I_2(3) \times \mathbb{Z}/2\mathbb{Z}, \qquad \underbrace{I_2(3)}_{\cong\, \mathfrak{S}_3}$$

$I_2(6)$ irreducible: $I_2(6) = W\big(\overset{6}{\underset{s_1 \quad s_2}{\circ\!-\!\!-\!\!-\!\circ}}\big)$

$I_2(3) \times \mathbb{Z}/2\mathbb{Z}$ reducible: $I_2(3) \times \mathbb{Z}/2\mathbb{Z} = W\big(\overset{3}{\underset{t_1 \quad t_2}{\circ\!-\!\!-\!\!-\!\circ}} \quad \underset{t_3}{\circ}\big)$





We'll often assume (W,S) is irreducible, but there is still another obvious way for the geom. rep'n $W \xrightarrow{\sigma} GL(V)$ to have a nontrivial W-stable subspace:

Whenever $B(\cdot,\cdot)$ on V is degenerate,
meaning $\{0\} \subsetneq \ker B := V^\perp$

$$= \{x \in V : B(x,\alpha) = 0 \ \forall \alpha \in V\}$$

since $V = \mathrm{span}_{\mathbb{R}}\{\alpha_1, \ldots, \alpha_n\}$

$$= \{x \in V : B(x,\alpha_i) = 0 \ \forall \alpha_i \in \Pi\}$$

$$= \bigcap_{i=1}^{n} V^{s_i}$$

since $W = \langle s_1, \ldots, s_n \rangle$

$$= V^W = \{x \in V : w(x) = x \ \forall w \in W\}$$

PROPOSITION: For (W,S) irreducible,

(a) Every proper W-stable subspace $V' \subsetneq V$ actually lies inside $V^\perp$, i.e. $V' \subseteq V^\perp$.

In particular,

(b) when $B(\cdot,\cdot)$ is degenerate, $W \xrightarrow{\sigma} GL(V)$ is not completely reducible, because one cannot have a W-stable complement U for $V^\perp$, i.e. $V \neq V^\perp \oplus U$.

(c) and when $B(\cdot,\cdot)$ is non-degenerate, $W \xrightarrow{\sigma} GL(V)$ is irreducible.

proof: When (W,S) has connected Cox. diagram, we claim any proper, W-stable subspace $V' \subsetneqq V$ cannot contain any of the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$:

otherwise if $V' \ni \alpha_i$, it will also contain any α_j for which $\underset{s_i}{\circ} \overset{m_{ij}}{—\!—} \underset{s_j}{\circ}$ with $m_{ij} \neq 2$

via the calculation

$$V' \ni s_j(\alpha_i) = \alpha_i - 2\underbrace{B(\alpha_i, \alpha_j)}_{= -\cos\left(\frac{\pi}{m_{ij}}\right) \neq 0} \cdot \alpha_j .$$

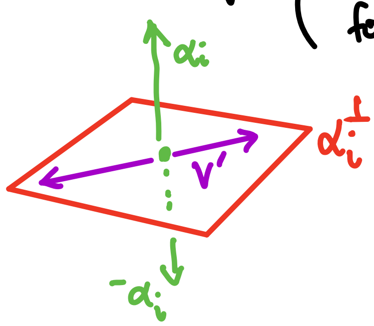
Hence $V' \supset \{\alpha_1, \ldots, \alpha_n\}$ so $V' = V$.

But now we claim $V' \subset V^{s_i} = \alpha_i^{\perp}$ for each $i = 1, 2, \ldots, n$ (and hence $V' \subset \bigcap_{i=1}^{n} V^{s_i} = V^{\perp}$) by the following reasoning.

When s_i acts on V' it acts semisimply (its minimal polynomial on V' divides its min poly on V, which is $x^2 - 1 = (x-1)(x+1)$),

$$\begin{array}{ccccc} \text{so } V & = & \alpha_i^{\perp} & \oplus & \mathbb{R}\alpha_i \\ \cup & & \cup & & \cup \\ V' & = & \begin{pmatrix}\text{1-eigenspace} \\ \text{for } s_i\end{pmatrix} & \oplus & \boxed{\begin{pmatrix}\text{(-1)-eigenspace} \\ \text{for } s_i\end{pmatrix}} \end{array}$$

must be $\{0\}$, since $\alpha_i \notin V'$.



[(b), (c) are direct from (a)] ▨

COROLLARY: When (W,S) is irreducible, the only $\varphi \in \mathrm{End}(V)$ that commute with the action of $W \xrightarrow{\sigma} GL(V)$ are the scalars $\varphi = c \cdot 1_V$, $c \in \mathbb{R}$.

NB: We can't just apply Schur's lemma — V may not be an irreducible W-rep'n, and $\mathbb{R}$ is not algebraically closed.

proof: Given $\varphi \in \mathrm{End}(V)$ commuting with W, pick any $s_i \in S$, and then φ acts on the line $\mathbb{R}\alpha_i$, since $\varphi(\alpha_i)$ lies in the (-1)-eigenspace for s_i:

$$s_i(\varphi(\alpha_i)) = \varphi(s_i(\alpha_i)) = \varphi(-\alpha_i) = -\varphi(\alpha_i).$$

Then φ scales $\mathbb{R}\alpha_i$ by some $c \in \mathbb{R}$, and we claim $\varphi = c \cdot 1_V$ since $V' = \ker(\varphi - c \cdot 1_V) \subsetneq V$ is a W-stable subspace, but can't lie inside $V^{\perp}$ since it contains $\mathbb{R}\alpha_i$.

EXERCISE: Why?

So $V' = \{0\}$ by PROP (a) above, meaning $\varphi = c \cdot 1_V$ ∎

To understand more about when (W,S) has W finite we need a tiny bit of (point-set) topology coming from...

The chambers and Tits cone inside V^*

Aspects of the geometry for how W and W_J act are easier to follow in V^* than in V!

Given (W,S), we created V with $\mathbb{R}$-basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and $B(\cdot,\cdot)$ to make the geom. rep'n $W \xrightarrow{\sigma} GL(V)$

Now create $V^* = \{\mathbb{R}\text{-linear functionals } f: V \to \mathbb{R}\}$ with dual basis $\{f_1, \ldots, f_n\}$ (i.e. $f_i(\alpha_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else.} \end{cases}$) to make the ~~contragredient~~ dual geom. rep'n. $W \xrightarrow{\sigma^*} GL(V^*)$

$$w \longmapsto \sigma^*(w)(f) = f \circ \bar{w}^{1}, \quad V \xrightarrow{w^{-1}} V \xrightarrow{f} \mathbb{R}$$

In matrix terms,

If $\sigma(w) = \begin{array}{c|c} & \alpha_1 \cdots \alpha_n \\ \hline \begin{array}{c}\alpha_1\\ \vdots \\ \alpha_n\end{array} & \begin{bmatrix} A(w) \end{bmatrix} \end{array}$ then $\sigma^*(w) = \begin{array}{c|c} & f_1 \cdots f_n \\ \hline \begin{array}{c}f_1\\ \vdots \\ f_n\end{array} & \begin{bmatrix} A(w^{-1})^T \end{bmatrix} \end{array}$

DEF'N: The (open) fundamental chamber $C \subset V^*$

$C := \mathbb{R}_{>0} f_1 + \ldots + \mathbb{R}_{>0} f_n = \{f \in V^* : f(\alpha_i) > 0 \ \forall \alpha_i \in \Pi\} = \{f \in V^* : f(\beta) > 0 \ \forall \beta \in \Phi^+\}$

Its closure is

$D := \overline{C} := \mathbb{R}_{\geq 0} f_1 + \ldots + \mathbb{R}_{\geq 0} f_n = \{f \in V^* : f(\alpha_i) \geq 0 \ \forall \alpha_i \in \Pi\}$

The Tits cone $\mathcal{U} := \bigcup_{w \in W} w(D) \subset V^*$

EXAMPLES (1) $W = \mathfrak{S}_3 = I_2(3) = W(\underset{s_1}{\circ} \overset{3}{\text{—}} \underset{s_2}{\circ})$

$V = \mathbb{R}^2$ with basis $\{\alpha_1, \alpha_2\}$, $B(\alpha_1,\alpha_2) = -\cos(\pi/3) = -\frac{1}{2}$

and $\sigma: W \to GL(V)$ has

$$\sigma_1 = \begin{array}{c|c} & \alpha_1 \quad \alpha_2 \\ \hline \begin{matrix}\alpha_1\\ \alpha_2\end{matrix} & \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \sigma_2 = \begin{array}{c|c} & \alpha_1 \quad \alpha_2 \\ \hline \begin{matrix}\alpha_1\\ \alpha_2\end{matrix} & \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \end{array}$$

$\left[\begin{array}{l}\text{since, e.g.,}\\ \sigma_2(\alpha_1) = \\ \alpha_1 - 2B(\alpha_1,\alpha_2)\,\alpha_2 = \\ \alpha_1 - 2\left(-\frac{1}{2}\right)\alpha_2 = \\ \alpha_1 + \alpha_2\end{array}\right]$

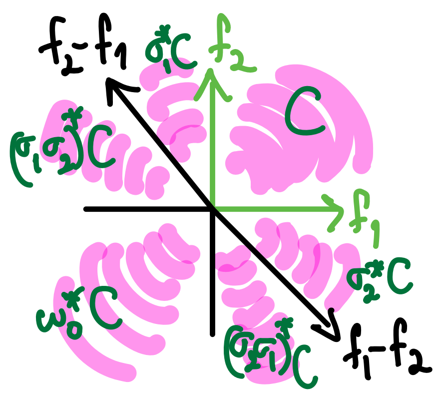
$$\Rightarrow \quad \sigma_1^* = \begin{array}{c|c} & f_1 \quad f_2 \\ \hline \begin{matrix}f_1\\ f_2\end{matrix} & \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \qquad \sigma_2^* = \begin{array}{c|c} & f_1 \quad f_2 \\ \hline \begin{matrix}f_1\\ f_2\end{matrix} & \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \end{array}$$

$$(\sigma_1\sigma_2)^* = \sigma_2^*\sigma_1^* = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{array}{c|c} & f_1 \quad f_2 \\ \hline \begin{matrix}f_1\\ f_2\end{matrix} & \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \end{array}$$

$$(\sigma_2\sigma_1)^* = \sigma_1^*\sigma_2^* = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{array}{c|c} & f_1 \quad f_2 \\ \hline \begin{matrix}f_1\\ f_2\end{matrix} & \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \end{array}$$

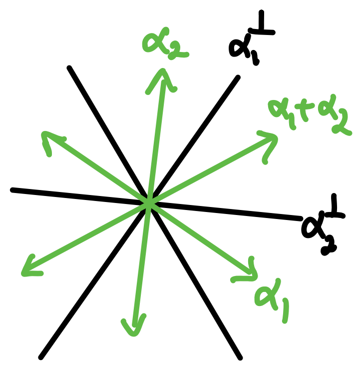
$$\sigma_1^*\sigma_2^*\sigma_1^* = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{array}{c|c} & f_1 \quad f_2 \\ \hline \begin{matrix}f_1\\ f_2\end{matrix} & \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \end{array} = \sigma_2^*\sigma_1^*\sigma_2^* = w_0^*$$

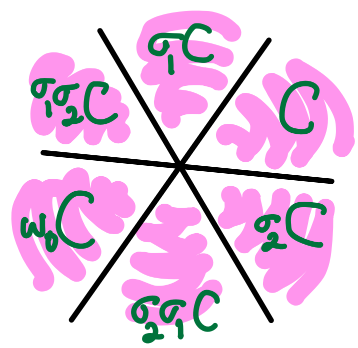
Picture in V^*:



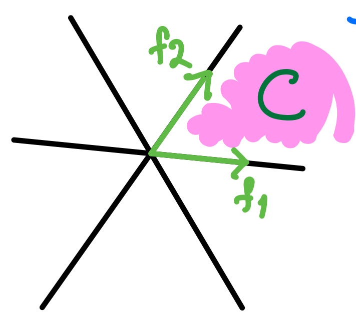
$\mathcal{U}$ = all of V^*

Compare with pictures in V





which looks pretty similar if we identify $V \cong V^*$ via $B(\cdot,\cdot)$



(2) $W = I_2(\infty) = W\big(\underset{s_1}{\circ}\overset{\infty}{\text{—}}\underset{s_2}{\circ}\big)$

$V = \mathbb{R}^2$ with basis $\{\alpha_1, \alpha_2\}$, $B(\alpha_1,\alpha_2) = -\cos\left(\frac{\pi}{\infty}\right) = -1$

and $W \xrightarrow{\sigma} GL(V)$ has

$$\sigma_1 = \begin{array}{c|cc} & \alpha_1 & \alpha_2 \\ \hline \alpha_1 & -1 & 2 \\ \alpha_2 & 0 & 1 \end{array} \qquad \sigma_2 = \begin{array}{c|cc} & \alpha_1 & \alpha_2 \\ \hline \alpha_1 & 1 & 0 \\ \alpha_2 & 2 & -1 \end{array}$$

[e.g. $\sigma_2(\alpha_1) = \alpha_1 - 2(-1)\alpha_2 = \alpha_1 + 2\alpha_2$]

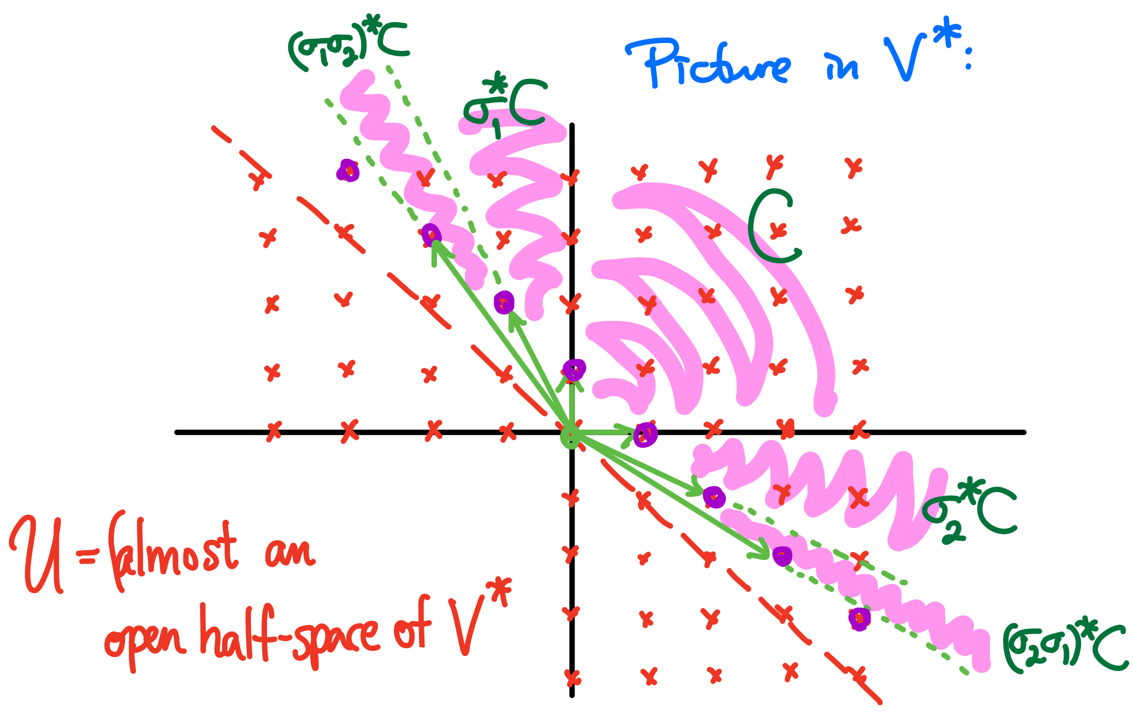
$$\Rightarrow \quad \sigma_1^* = \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & -1 & 0 \\ f_2 & 2 & 1 \end{array} \qquad \sigma_2^* = \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 1 & 2 \\ f_2 & 0 & -1 \end{array}$$

$$(\sigma_1\sigma_2)^* = \sigma_2^*\sigma_1^* = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 3 & 2 \\ f_2 & -2 & -1 \end{array}$$

$$(\sigma_2\sigma_1)^* = \sigma_1^*\sigma_2^* = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & -1 & -2 \\ f_2 & 2 & 3 \end{array}$$

$$(\sigma_1\sigma_2\sigma_1)^* = \sigma_1^*\sigma_2^*\sigma_1^* = \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & -3 & -2 \\ f_2 & 4 & 3 \end{array} \qquad (\sigma_2\sigma_1\sigma_2)^* = \sigma_2^*\sigma_1^*\sigma_2^* = \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 3 & 4 \\ f_2 & -2 & -3 \end{array}$$

$\vdots \qquad\qquad \vdots$



These chambers $w(C)$ $\left(:= \sigma^*(w)(C)\right)$ inside the Tits cone in V^* let us reinterpret yet again

$$T_L(w) := \{t \in T : \ell(tw) < \ell(w)\}$$ left-associated reflections of w

$$= \{t_\beta : \beta \in \Phi^+ \text{ and } w^{-1}(\beta) \in \Phi^-\}$$

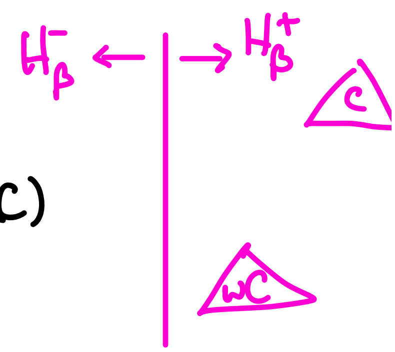
Every $\beta \in \Phi^+$ disjointly decomposes V^* into

a hyperplane $H_\beta^\circ := \{f \in V^* : f(\beta) = 0\}$

and two open halfspaces $H_\beta^+ := \{f \in V^* : f(\beta) > 0\}$ $(\supset C \ \forall \beta \in \Phi^+)$

$$H^- := \{f \in V^* : f(\beta) < 0\}$$

PROPOSITION: For any $w \in W$ and $\beta \in \Phi^+$,

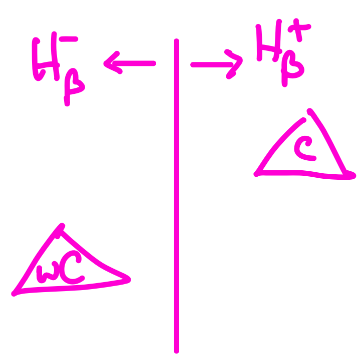


$$\ell(t_\beta w) > \ell(w) \iff w(C) \subset H_\beta^+$$

i.e. H_β° does not separate C from $w(C)$

$$\ell(t_\beta w) < \ell(w) \iff w(C) \subset H_\beta^-$$

i.e. H_β° separates C from $w(C)$



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proof: Translate:

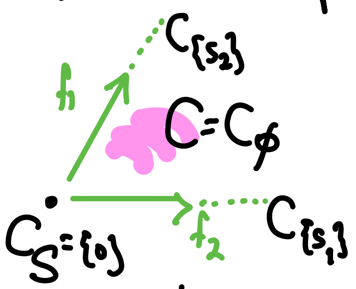
$\ell(t_\beta w) < \ell(w) \Rightarrow w^{-1}(\beta) \in \Phi^-$

$\Rightarrow f(w^{-1}(\beta)) < 0$ for $f \in C$

$\quad \| \ (f \circ w^{-1})(\beta) = \sigma^*(w)(f)(\beta)$ i.e. $w(C) \subset H_\beta^-$ ∎

This helps us disjointly decompose the Tits cone into faces and understand their stabilizers.

First (disjointly) decompose $D = \overline{C} = \bigsqcup_{J \subseteq S} C_J$



where $C_J := \bigcap_{s_j \in J} H^{\circ}_{\alpha_j} \cap \bigcap_{s_i \in S \setminus J} H^{+}_{\alpha_i}$

and then note this implies a (not necessarily disjoint) covering $\mathcal{U} := \bigcup_{w \in W} wD = \bigcup_{\substack{J \subseteq S \\ w \in W}} wC_J$

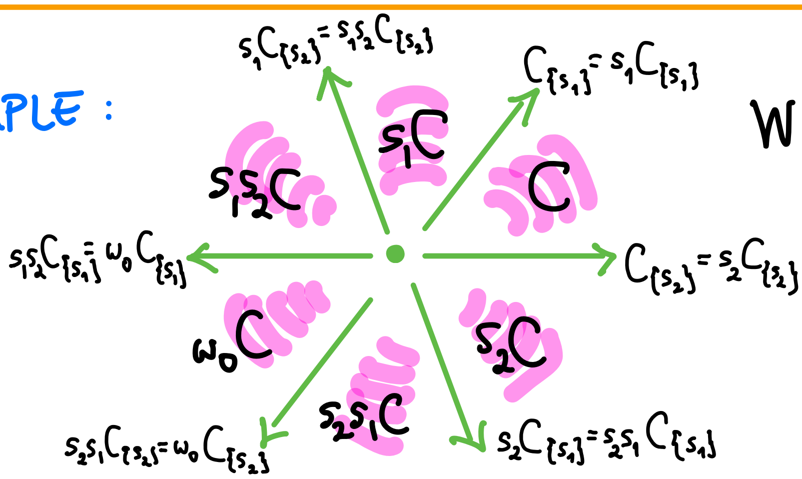
THEOREM: $\mathcal{H}$ is disjoint:

$$wC_J \cap w'C_{J'} \neq \phi \iff J = J' \text{ and } wC_J = w'C_J \text{ and } wW_J = w'W_J$$

Furthermore, D is a fundamental domain for the W-action on $\mathcal{U}$: every $f \in \mathcal{U}$ has a unique image $w(f) \in D$.

EXAMPLE:

$W(\overset{3}{\underset{s_1}{\circ} - \underset{s_2}{\circ}}) = \mathfrak{S}_3$



proof: When $wC_J \cap w'C_{J'} \neq \emptyset$, reduce to the case where $w'=1$ by multiplying on left by w'^{-1}, i.e. need to show

$$wC_J \cap C_{J'} \neq \emptyset \iff J=J' \text{ and } w\in W_J \quad (\text{so } wC_J = C_J = C_{J'}.)$$

The backward implication ($\Leftarrow$) is easy, and for the forward ($\Rightarrow$), proceed by induction on $\ell(w)$.

Write $w = s_i \cdot s_i w$ for some s_i in S with $\ell(s_i w) < \ell(w)$.

By the previous PROP, $wC \subset H^-_{\alpha_i}$

continuity of $\sigma(w): V^* \to V^*$

$$\Rightarrow wD \subset \overline{H^-_{\alpha_i}}$$

closure

Since $D \subset \overline{H^+_{\alpha_i}}$, one concludes

$$\begin{array}{c} wD \cap D \subset \overline{H^+_{\alpha_i}} \cap \overline{H^-_{\alpha_i}} = H^0_{\alpha_i} = \{f \in V^* : s_i(f) = f\} \\ \cup \\ wC_J \cap C_{J'} \ (\neq \emptyset) \end{array}$$

Pick some $f \in wC_J \cap C_{J'}$, and since $f \in C_{J'}$ and $s_i(f)=f$, one concludes $s_i \in J'$. But also, applying s_i, one has

$$s_i w C_J \cap s_i C_{J'} \neq \emptyset$$
$$= s_i w C_J \cap C_{J'},$$

so induction applies to give $J=J'$ and $s_i w \in W_J = W_{J'}$.

And then also $w \in W_J$

For the furthermore statement, since $\mathcal{U} := \bigcup_{w\in W} wD$, every $f \in \mathcal{U}$ has some image $w(f) \in D$. If it had two of them, say $w(f) = w'(f) \in D = \bigsqcup_{J \subseteq S} C_J$,

$\exists$ some J, J' with $w(f) \in C_J$ i.e. $f \in w^{-1} C_J$
$\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad w'(f) \in C_{J'}$ i.e. $f \in (w')^{-1} C_{J'}$

so $J = J'$ and $w^{-1} W_J = (w')^{-1} W_J$ and $w(f) = w'(f)$ $\boxtimes$

This has a nontrivial consequence about the induced topology of $\sigma^*(W) \subset GL(V^*) \subset \mathbb{R}^{n^2}$ and $\sigma(W) \subset GL(V) \subset \mathbb{R}^{n^2}$

COROLLARY: Both $\sigma^*(W), \sigma(W)$ carry the discrete topology as subsets of $\mathbb{R}^{n^2}$, even if W is infinite.

$\forall w \in W$ $\exists$ an open subset $\mathcal{O}_w \subset \mathbb{R}^{n^2}$ with $\mathcal{O}_w \cap \sigma^*(W) = \{\sigma^* w\}$.

proof: Enough to show it for $\sigma^*(W)$, since $\begin{array}{ccc} \mathbb{R}^{n^2} & \longrightarrow & \mathbb{R}^{n^2} \\ A & \longmapsto & A^T \end{array}$

is a homeomorphism.

Fix $f_0 \in C$ = open fundamental chamber in V^*

Then wC is an open neighborhood of $w(f_0)$ in $V^* (= \mathbb{R}^n)$.

Consider the map
$$\begin{aligned} GL(V^*) &\xrightarrow{\varphi} V^* \\ g &\longmapsto g(f_0) \end{aligned}$$
which is continuous, since if $g = (g_{ij})$ and $f_0 = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

then $g(f_0) = \begin{bmatrix} \sum_j g_{1j} a_j \\ \vdots \\ \sum_j g_{nj} a_j \end{bmatrix}$.

Hence $\Theta_w := \varphi^{-1}(wC)$ is an open subset in $GL(V^*) \subset \mathbb{R}^{n^2}$

and one has $\varphi^{-1}(wC) \cap \sigma^*(W) = \{\sigma^*(w)\}$

since $v \in W$ has $\sigma^*(v) \in \varphi^{-1}(wC)$
$$\begin{aligned} &\iff \varphi(\sigma^*(v)) \in wC \\ &\qquad \varphi(\sigma^*(v)) = \sigma^*(v)(f_0) = v(f_0) \\ &\iff v = w \quad \text{by previous THEOREM} \quad \blacksquare \end{aligned}$$

This finally allows us to tie finiteness of W in (W,S) with positive definiteness of $B(\cdot,\cdot)$ (and finite real ref'n groups).

THEOREM: For a Coxeter system (W,S), T.F.A.E.:

(a) The bilinear form $B(\cdot,\cdot)$ on V is positive definite.

(b) W is finite.

(c) W acts on V via the geom. rep'n $W \xrightarrow{\sigma} GL(V)$ as a (finite) real ref'n group.

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proof: It's enough to show (a) $\Leftrightarrow$ (b), since (c) $\Rightarrow$ (b), and if we know (a) $\Rightarrow$ (b) then we also know (a) $\Rightarrow$ (c).

For (a) $\Rightarrow$ (b), note that since $B(\cdot,\cdot)$ is positive definite on V, one can find an orthonormal basis $e_1, \ldots, e_n$ for V via Gram-Schmidt, and so

$$W \xrightarrow{\sigma} O_n(\mathbb{R}) \subset GL(V)$$

$$O_n(\mathbb{R}) = \{\text{orthogonal matrices } A \text{ in } \mathbb{R}^{n\times n} \text{ i.e. } A^T A = I_n\}$$

However $O_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ is compact, because it is

- closed (defined by the equations $A^T A = I_n$)
- bounded (each column vector v_i has $\|v_i\|^2 = v_i^T v_i = 1$, so $O_n(\mathbb{R})$ lies inside sphere of radius $\sqrt{n}$)

Recall the subset $\sigma(W) \subset O_n(\mathbb{R})$ has the discrete topology by a previous COROLLARY.

We claim this forces $\sigma(W)$ to be finite by the following reasoning, using the discreteness twice:

- $\sigma(W)$ is closed inside $O_n(\mathbb{R})$ by a general...

LEMMA: Discrete subgroups of Hausdorff topological groups are closed.

proof: see Math Stack Exchange #29515 "Why is every discrete subgroup of a Hausdorff group closed?" and the answer by user TwoTwo for a relatively quick proof. ▨

omitted this in lecture 10/21/2022

- if $\sigma(W)$ were infinite, it would have an accumulation point g_0 in the compact set $O_n(\mathbb{R})$, and g_0 would also lie in $\sigma(W)$ by its closure, which would violate $\sigma(W)$'s discrete topology.

Thus $\sigma(W)$ is finite, and hence W is finite,

since $W \xrightarrow{\sigma} GL(V)$ is faithful.

For (b) $\Rightarrow$ (a), assume W is finite.

WLOG we can assume (W,S) is irreducible,

since otherwise $W = W_1 \times W_2 \times \ldots \times W_m$

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_m$$

orthogonal direct sum for $B(\cdot,\cdot)$

Then by Maschke's Theorem, $W \xrightarrow{\sigma} GL(V)$ is completely reducible since W is finite, and a prior result tells us that

$$\begin{cases} B(\cdot,\cdot) \text{ is nondegenerate on } V, \text{ and} \\ \text{The only } \varphi : V \to V \text{ commuting with } W \\ \quad \text{are scalars } \varphi = c \cdot 1_V \end{cases}$$

We want to compare $B(\cdot,\cdot)$ with a positive definite W-invariant form on V, which we can create as follows:

- Start with any positive definite form $B'(\cdot,\cdot)$ on V,

  e.g. $B'(x,y) := \sum_{i=1}^{n} x_i y_i$ if $x = \sum_{i=1}^{n} x_i e_i$, $y = \sum_{j=1}^{n} y_j e_j$

  for some basis $e_1, \ldots, e_n$ of V

- Average it to get one that is W-invariant:

Define $B''(x,y) := \frac{1}{\#W} \sum_{w \in W} B'(w(x), w(y))$

Easy Checks: $\left\{ \begin{array}{l} B'' \text{ is } \mathbb{R}\text{-bilinear, symmetric because } B' \text{ is} \\ B'' \text{ is pos. def. because } B' \text{ is} \\ B'' \text{ is } W\text{-invariant, i.e. } B''(v(x),v(y)) = B''(x,y) \end{array} \right.$

Now we claim that $\exists\, c \in \mathbb{R} - \{0\}$ with $B(x,y) = c\, B''(x,y)$, because the composite of two $\mathbb{R}$-linear isomorphisms

$$\begin{array}{ccccc} V & \xrightarrow{\sim} & V^* & \xleftarrow{\sim} & V \\ x & \longmapsto & B(x,-) & & \\ & & B''(y,-) & \longleftarrow & y \end{array}$$

leads an $\mathbb{R}$-linear isomorphism $V \xrightarrow{\varphi} V$ that commutes with $W \xrightarrow{\sigma} GL(V)$:

if $\varphi(x)=y$, so that $B(x,-)=B''(y,-)$

meaning $B(x,z)=B''(y,z)$ $\forall z\in V$

then for any $w\in W$ one has $\varphi(w(x))=w(y)$ since

$$
\begin{aligned}
B(w(x),z) &= B(w^{-1}w(x), w^{-1}(z)) \quad \text{W-invariance of } B(\cdot,\cdot) \\
&= B(x, w^{-1}(z)) \\
&= B''(y, w^{-1}(z)) \\
&= B''(w(y), ww^{-1}(z)) \quad \text{W-invariance of } B''(\cdot,\cdot) \\
&= B''(w(y), z).
\end{aligned}
$$

Once one knows $B(x,y)=c\cdot B''(x,y)$, one knows $c>0$

since $B(\alpha,\alpha)=1$

$B''(\alpha,\alpha)>0$.

Hence $B(\cdot,\cdot)$ is also positive definite, since $B''(\cdot,\cdot)$ is.

$\blacksquare$