Classifications

GOAL: Classify Cox. systems (W,S) with W finite

(= finite real ref'n groups
= those (W,S) with $B(\cdot,\cdot)$ positive definite)

OBVIOUS REDUCTION:
Assume (W,S) irreducible, i.e. Cox diagram connected.

LESS OBVIOUS:
It helps to track the ones with this extra property:

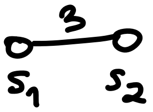
DEF'N: W a finite ref'n group acting on $V=\mathbb{R}^n$ is called crystallographic or a Weyl group if it stabilizes some rank n lattice $L \subset \mathbb{R}^n$

i.e. $L \cong \mathbb{Z}^n$ as groups

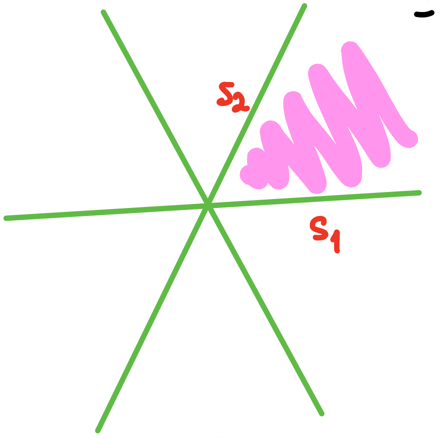
The Weyl groups W turn out be the ones that have an associated affine ref'n group $\tilde{W}$, which turn out to be the $(\tilde{W},\tilde{S})$ where $\tilde{B}(\cdot,\cdot)$ is positive semidefinite, but degenerate.

(... and Weyl groups come from semisimple Lie groups)

EXAMPLES

(1) $W\Bigg($ 

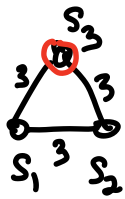
$\Bigg) = W = \mathfrak{S}_3 = A_2$



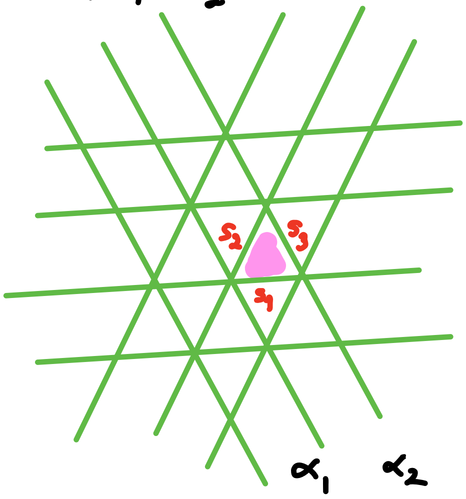
Gram matrix for $B(\cdot,\cdot)$:

$$\begin{array}{c|cc} & \alpha_1 & \alpha_2 \\ \hline \alpha_1 & 1 & -\frac{1}{2} \\ \alpha_2 & -\frac{1}{2} & 1 \end{array}$$

pos. def.

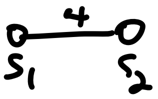
$W\Bigg($ 

$\Bigg) = \tilde{W} = \tilde{\mathfrak{S}}_3$

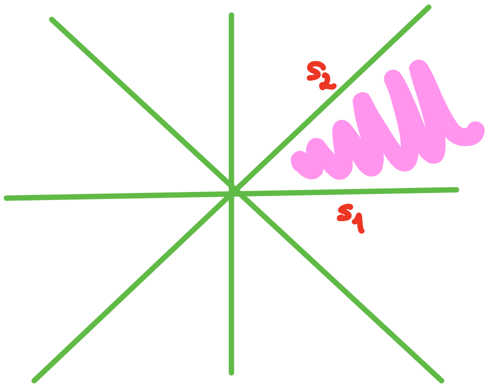


$$\begin{array}{c|ccc} & \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ \alpha_2 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \alpha_3 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}$$

pos. semidef. (det = 0)

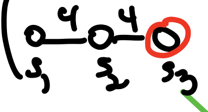
(2) $W\Big($ 

$\Big) = W = B_2 = C_2$

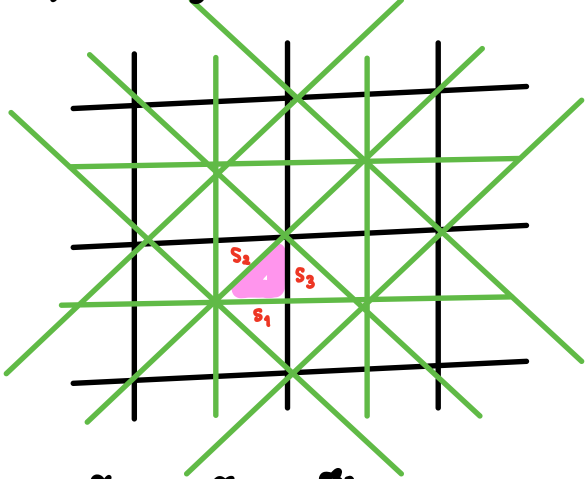


$$\begin{array}{c|cc} & \alpha_1 & \alpha_2 \\ \hline \alpha_1 & 1 & -\frac{\sqrt{2}}{2} \\ \alpha_2 & -\frac{\sqrt{2}}{2} & 1 \end{array}$$

pos. def.

$W\Big($ 

$\Big) = \tilde{W} = \tilde{B}_2 = \tilde{C}_2$



$$\begin{array}{c|ccc} & \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_1 & 1 & -\sqrt{2}/2 & 0 \\ \alpha_2 & -\sqrt{2}/2 & 1 & -\sqrt{2}/2 \\ \alpha_3 & 0 & -\sqrt{2}/2 & 1 \end{array}$$

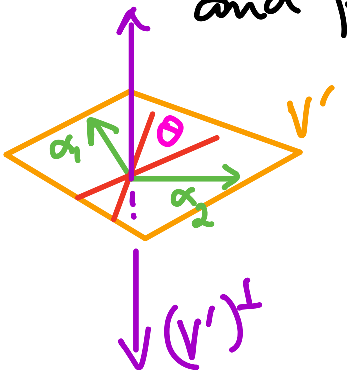
pos. semidef. (det = 0)

Here's an easy fact that helps spot Weyl groups.

PROPOSITION: If (W,S) is a Weyl group then all $m_{ij} \in \{2,3,4,6\}$.

proof: In the geom. rep'n $W \xrightarrow{\sigma} GL(V)$ we could pick a $\mathbb{Z}$-lattice basis $v_1, \dots, v_n$ for L as our $\mathbb{R}$-basis for V, to write matrices for the action of w. So $\text{trace}(\sigma(w)) \in \mathbb{Z}$ $\forall w \in W$.

Apply this to $s_i s_j$, which acts on the 2-plane $V' = \text{span}_{\mathbb{R}}\{\alpha_i, \alpha_j\}$ as rotation through $\theta = \frac{2\pi}{m_{ij}}$ and pointwise fixes $(V')^\perp$, one has $\sigma(w)$ acting as



$$\begin{array}{c|c|c} & V' & (V')^\perp \\ \hline V' & \begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix} & 0 \\ \hline (V')^\perp & 0 & \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{matrix} \end{array}$$

with trace $(n-2) + 2\cos\theta$.

$$\Rightarrow 2\cos\theta \in \mathbb{Z}$$

$$\cos\theta \in \tfrac{1}{2}\mathbb{Z}$$

Since $\theta \in (0, \pi]$, this forces $\cos\theta \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}\}$

$$\Downarrow$$

$m_{ij} \in \{2,3,4,6\}$ ▨

THE IRREDUCIBLE FINITES & AFFINES:

Since $m_{ij}=3$ are so common among finite/affine (W,S), let's omit those edge labels:

We'll do the infinite families first.

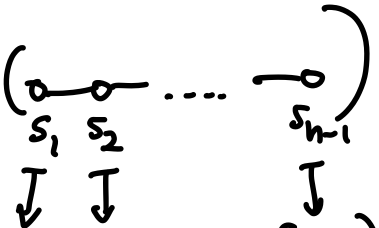
We've met...

Cartan-Killing names from Lie theory

TYPE A

symmetric group

$W = \mathfrak{S}_n = W(A_{n-1}) = W\Big($



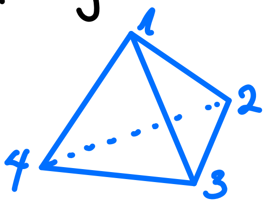
$\Big)$

$n \geq 2$

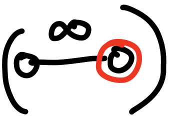
$\left[\begin{array}{c|c} \begin{matrix}0&1\\1&0\end{matrix} & 0 \\ \hline 0 & \begin{matrix}1&&0\\&\ddots&\\0&&1\end{matrix}\end{array}\right] = (1,2) \quad (2,3) \quad (n-1,n)$

= Weyl group (of Lie algebra $\mathfrak{sl}_n$)

= symmetries of regular $(n-1)$-simplex

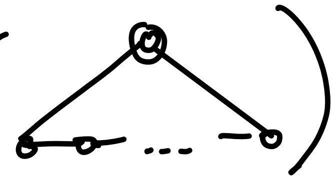


$\tilde{W} = \tilde{\mathfrak{S}}_n = \Big\{\; I_2(\infty) = W\Big($



$\Big)$ if $n=2$

$W\Big($



$\Big)$ if $n \geq 3$

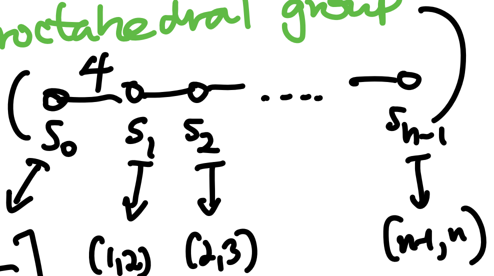
see Björner-Brenti §8.3

affine symmetric group

TYPE B/C

hyperoctahedral group

$W = W($



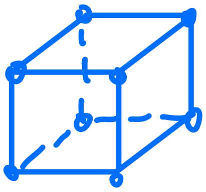
$) = W(B_n) = W(C_n)$

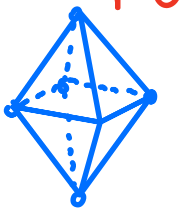
$n \geq 2$

$$\left[\begin{array}{c|ccc} -1 & & 0 & \\ \hline & 1 & & 0 \\ 0 & & \ddots & \\ & 0 & & 1 \end{array}\right]$$

= Weyl group of Lie algebras $\mathfrak{so}_{2n+1}$ & $\mathfrak{sp}_{2n}$

= symmetries of regular n-cube or n-cross-polytope



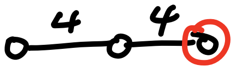


= signed permutation matrices

see Björner-Brenti §8.1

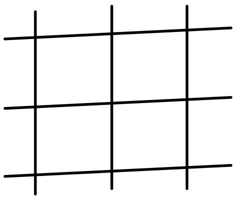
$\tilde{W} =$

$\tilde{B}_2 = \tilde{C}_2 = W($



$)$ if $n=2$

= symmetries of tesselation

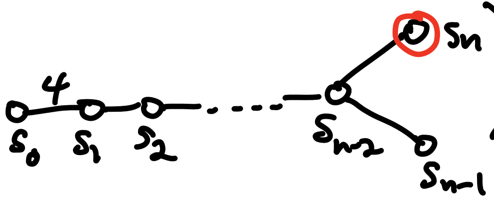


$\tilde{C}_n = W($

$)$

$n \geq 3$ = symmetries of n-cubical or n-cross-polytopal tesselation of $\mathbb{R}^n$

$\tilde{B}_n = W($



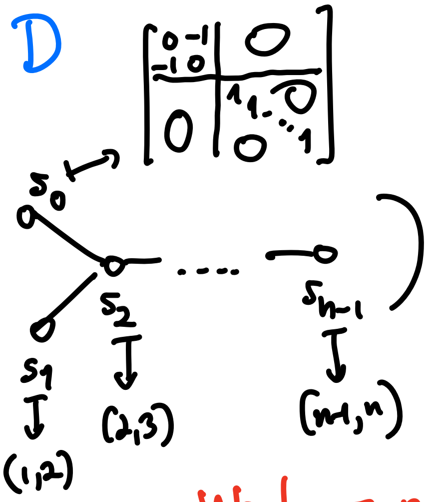
$)$

see Björner-Brenti §8.5

see Björner-Brenti §8.4

TYPE D

$W = W\Big($



$\Big) = W(D_n)$

$n \geq 4$

see Björner-Brenti §8.2

$= \text{Weyl group of Lie algebra } \mathfrak{so}_{2n}$

$=$ signed permutation matrices with an even number of minus signs, e.g.

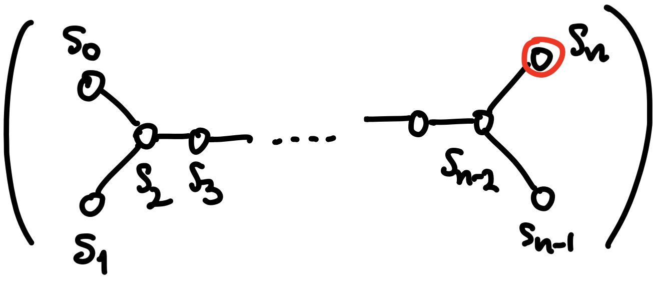
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \ker\left(W(B_n) \xrightarrow{\varphi} \{\pm 1\} \right)$$

$$w \longmapsto (\text{product of nonzero entries in matrix } w)$$

$W(B_n) = W(C_n) =$ signed permutation matrices

$\tilde{W} = W\Bigg($

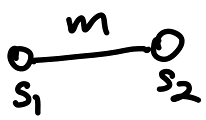


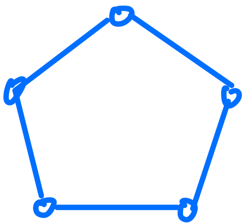
$\Bigg)$

see Björner-Brenti §8.6

Type I

dihedral group

W = W() = symmetries of regular m-gon

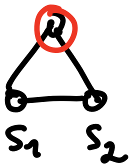
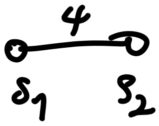
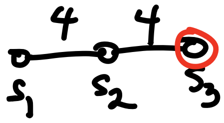
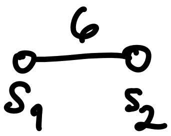
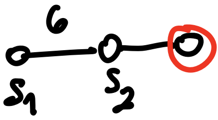
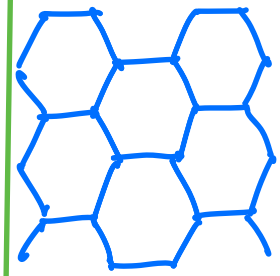
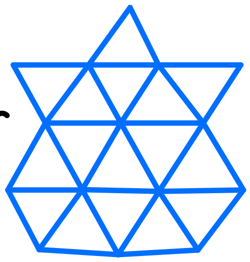


m=5

m ≥ 3

reducible

Not crystallographic/Weyl groups unless $m \in \{\cancel{2},3,4,6\}$:

m=	Weyl group W		affine Weyl group $\tilde{W}$
3		A_2	
4		$B_2 = C_2$	
6		G_2	 $\tilde{G}_2$ = symmetries of tessellations  or 

REMARK:

All of the infinite families $A_{n-1}, B_n = C_n, D_n, I_2$ of irreducible real ref'n groups W can be subsumed in one infinite family of complex ref'n groups

:= finite subgroups $G < GL(V)$, $V = \mathbb{C}^n$
DEF'N
gen'd by (complex) reflections
which are $t \in GL(V)$ with fixed space
$V^t = H$ a hyperplane
(but non-1 eigenvalue may be any complex root-of-unity, not necessarily -1)

DEF'N: The Shephard-Todd infinite family of imprimitive reflection groups $G(r,p,n)$ are defined by

$G(r,1,n) := \{ n \times n$ monomial matrices with nonzero entries all rth roots-of-unity $\}$

matrices with exactly one nonzero entry in each row and column e.g. $G(1,1,n) = W(A_{n-1}) = \mathfrak{S}_n$ = permutation matrices

$G(2,1,n) = W(B_n)$
$= W(C_n) =$ signed permutation matrices

and for general p dividing r,

$G(r,p,n) := \{$ matrices in $G(r,1,n)$ whose product of nonzero entries is an $(r/p)^{th}$ root-of-unity $\}$

$$= \ker\left(\begin{array}{ccc} G(r,1,n) & \xrightarrow{\varphi} & \left\{ \begin{array}{c} p^{th} \\ \text{roots-of-unity} \end{array} \right\} \\ A & \longmapsto & \left(\begin{array}{c} \text{product of} \\ \text{nonzero} \\ \text{entries of } A \end{array} \right)^{r/p} \end{array} \right)$$

e.g. $G(2,2,n) = W(D_n)$
$=$ signed permutation matrices with evenly many (-1)'s.

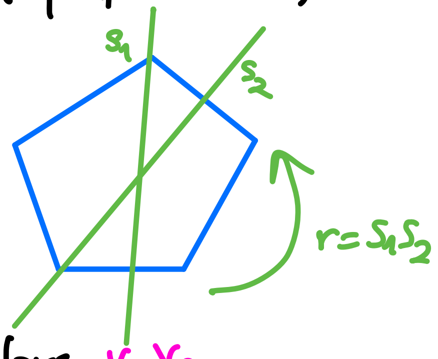
e.g. $G(m,2,2) \cong I_2(m)$

(!)

$= \langle s_1, r \mid s_1^2 = r^m = 1, \ s_1 r s_1^{-1} \rangle$

EXERCISE:

If you diagonalize the action of the rotation $r = s_1 s_2$ on $\mathbb{R}^2$



using $V = \mathbb{C}^2$, the two r-eigenvectors v_1, v_2 can be scaled so that s_1 swaps them $v_1 \overset{}{\underset{s_1}{\longleftrightarrow}} v_2$

This gives an isomorphism

$$I_2(m) \longrightarrow G(m,2,2)$$

$$r \longmapsto \begin{array}{c|cc} & v_1 & v_2 \\ \hline v_1 & \zeta & 0 \\ v_2 & 0 & \zeta^{-1} \end{array}$$

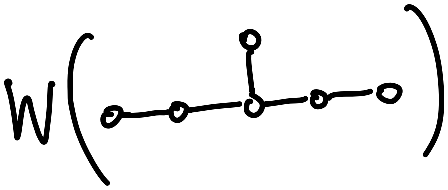
$$s_1 \longmapsto \begin{array}{c|cc} & v_1 & v_2 \\ \hline v_1 & 0 & 1 \\ v_2 & 1 & 0 \end{array}$$

Continuing with the classification ...

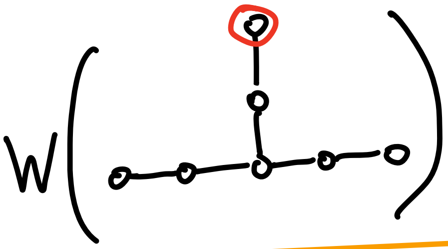
EXCEPTIONAL IRREDUCIBLE FINITE & AFFINES

TYPE E - all are Weyl groups, and simply-laced ($:=$ all $m_{ij} \in \{2,3\}$)

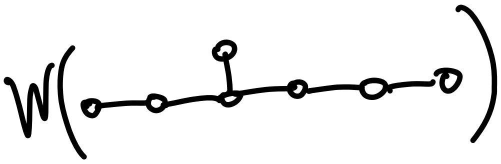
$W(E_6) =$



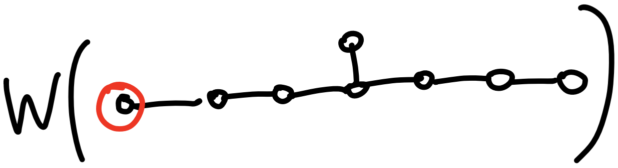
$\tilde{W}(E_6) =$



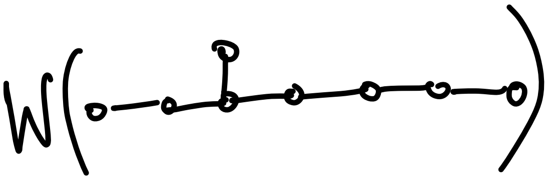
$W(E_7) =$



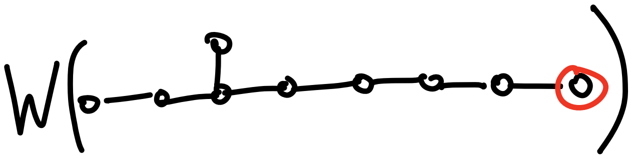
$\tilde{W}(E_7) =$



$W(E_8) =$



$\tilde{W}(E_8) =$



Only 3 of them!

TYPE F

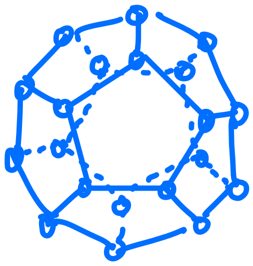
$W(F_4) = W(\circ - \circ \overset{4}{-} \circ - \circ)$ = symmetries of Schläfli's 24-cell, a 4-dimensional self-dual regular polytope

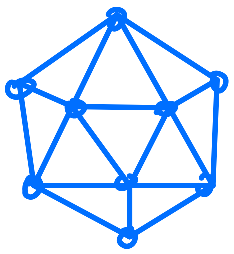
and also a Weyl group, with

$\widetilde{W}(F_4) = W(\circ - \circ \overset{4}{-} \circ - \circ - \circledcirc)$

TYPE H

$W(H_3) = W(\circ \overset{5}{-} \circ - \circ)$ = symmetries of dodecahedron & icosahedron





$W(H_4) = W(\circ \overset{5}{-} \circ - \circ - \circ)$ = symmetries of Schläfli's 120-cell & 600 cell, two 4-dimensional (dual) regular polytopes

(Neither one is a Weyl group.)

Classification ideas

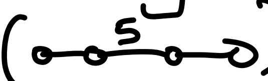
(1) It's not hard to check that all of the irreducible (W,S) on the finite lists above have $B(\cdot,\cdot)$ positive definite

(e.g., true for infinite families since they come from real refl'n groups we've seen, and for exceptionals can do a finite check of Gram matrices having positive eigenvalues, or positive northwest minor determinants).

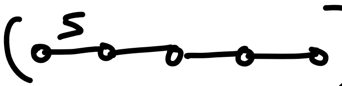
(2) It's also not hard to check all of those on the affine lists above are positive semidefinite but degenerate

(e.g. exhibit nullvectors, or compute determinants)

(3) After checking these two are indefinite

$W($ 

$)$ Z_4

$W($ 

$)$ Z_5

with negative Gram matrix determinants, one is ready to use this lemma ...

SUBDIAGRAM LEMMA: If an irreducible (W,S) has $B(\cdot,\cdot)$ pos. semidefinite, then its Coxeter diagram has every proper subdiagram (W',S') with $B'(\cdot,\cdot)$ pos. definite.

either decrease some labels $m'_{ij} < m_{ij}$, or omit some $s_i \in S$, or both.

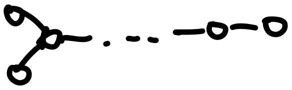
Strategy: Winnow down to these...

FINITE

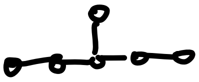
A

B/C

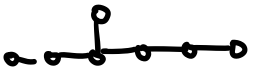
D



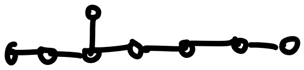
E_6



E_7

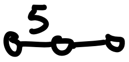


E_8

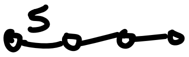


F_4

H_3



H_4

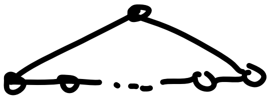


I_2

... because you can't have any of these as subdiagrams:

AFFINE

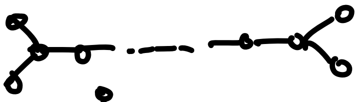
$\tilde{A}$



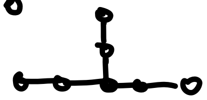
$\tilde{C}$

$\tilde{B}$

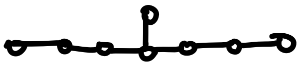
$\tilde{D}$



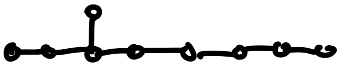
$\tilde{E}_6$



$\tilde{E}_7$

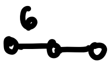


$\tilde{E}_8$



$\tilde{F}_4$

$\tilde{G}_2$



Z_4

Z_5

How to prove that SUBDIAGRAM LEMMA ?

First prove a useful Perron-Frobenius Lemma ...

LEMMA: Let $B=B^t$ be a real symmetric $n\times n$ positive definite matrix. Then

(a) $\ker B = \{x \in \mathbb{R}^n : x^t B x = 0\}$

If we furthermore assume B has

- $B_{ij} \leq 0 \quad \forall i \neq j$
- B is indecomposable, i.e. you can't decompose $\{1,2,\ldots,n\} = I \sqcup J$ with $b_{ij} = 0 \ \forall i \in I, j \in J$, $I, J \neq \phi$

then

(b) every $x \in \ker B - \{0\}$ has all coordinates of same sign, either $x_i > 0 \ \forall i$ or $x_i < 0 \ \forall i$, and $\dim_{\mathbb{R}}(\ker B) \leq 1$, and

(c) the smallest eigenvalue λ of B has multiplicity 1 and an all positive eigenvector x, i.e. $x_i > 0 \ \forall i$.

proof: (a): Diagonalize B orthogonally

so $B = P^{-1} D P$ where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $\lambda_i \geq 0$

$= P^t D P$

Then
$$0 = x^t B x = x^t P^t D P x$$
$$= (Px)^t D Px$$
$$= y^t D y = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2$$

let $y = Px$

$\iff y_i = 0$ for all $\lambda_i > 0$

$\iff Dy = \underline{0}$

$\iff DPx = \underline{0}$

$\iff P^t D P x = \underline{0}$ since P is invertible

$\iff Bx = \underline{0}$

(b): If $B_{ij} \leq 0$ for $i \neq j$, then for any $x \in \ker B$

we claim $|x| = \begin{bmatrix} |x_1| \\ \vdots \\ |x_n| \end{bmatrix} \in \ker B$ also:

$$0 \leq |x|^T B |x| = \sum_{i,j=1}^{n} B_{ij} |x_i| |x_j|$$
$$= \sum_{i=1}^{n} B_{ii} |x_i|^2 + \sum_{\substack{(i,j): \\ i \neq j}} \underbrace{B_{ij}}_{\leq 0} |x_i| |x_j| \leq \sum_{i,j} B_{ij} x_i x_j$$
$$= x^t B x = 0$$

But then if B is indecomposable, we further claim $|x|$ has all positive (nonzero) coordinates.

Otherwise, write $\{1,2,\ldots,n\} = I \sqcup J$

$$\underbrace{\underset{\underset{\{i:|x_i|=0\}}{\shortparallel}}{I} \qquad \underset{\underset{\{j:|x_j|>0\}}{\shortparallel}}{J}}_{\text{both non-empty}}$$

and $B|x| = \underline{0}$

$$\Rightarrow (B|x|)_i = 0 \quad \forall i \in I$$

$$\shortparallel$$

$$\sum_{j\in J} \underbrace{B_{ij}}_{\leq 0} \underbrace{|x_j|}_{>0}$$

$$\Rightarrow B_{ij} = 0 \quad \forall i \in I, j \in J \quad \lightning$$

This also shows any $x \in \ker B$ has all $x_i \neq 0$ (since $|x_i| > 0$)

and hence $\dim \ker B \leq 1$:

if $x, x' \in \ker B$ were lin. indep., could create $x'' = cx + c'x' \in \ker B$ with $x''_1 = 0$.

(c): If B has smallest eigenvalue λ, then $B' := B - \lambda I_n$ satisfies same hypotheses, and so $\ker(B') = \lambda$-eigenspace for B has dim 1, spanned by some $x = |x|$ with all $x_i > 0$

▨

This lets us deduce...

SUBDIAGRAM LEMMA: If an irreducible (W,S) has $B(\cdot,\cdot)$ pos. semidefinite, then its Coxeter diagram has every proper subdiagram (W',S') with $B'(\cdot,\cdot)$ pos. definite.

either decrease some labels $m'_{ij} < m_{ij}$, or omit some $s_i \in S$, or both.

proof: Let B, B' be their Gram matrices of sizes $n \geq n'$, and assume B' is not pos. definite.

So $\exists$ some $x \in \mathbb{R}^{n'} - \{\underline{0}\}$ with $x^t B' x \leq 0$,

and we can create $y = (|x_1|, \ldots, |x_{n'}|, 0, \ldots, 0) \in \mathbb{R}^n$ with

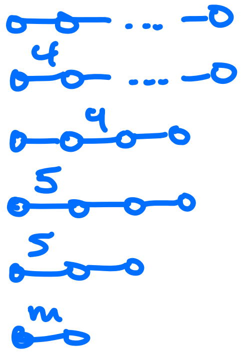
$$0 \leq y^t B y = \sum_{i,j=1}^{n'} \underset{\substack{\| \\ -\cos\left(\frac{\pi}{m_{ij}}\right) \leq 0}}{B_{ij}} \underbrace{|x_i||x_j|}_{\geq 0} \leq \sum_{i,j=1}^{n'} \underset{\substack{\| \\ -\cos\left(\frac{\pi}{m'_{ij}}\right) \leq 0}}{B'_{ij}} |x_i||x_j| \leq x^t B' x \leq 0$$

$\leq$ since $m_{ij} \geq m'_{ij}$

Hence equality holds throughout, and $y \in \ker B$.
This implies $y = |y|$ has all positive coordinates by PF Lemma.
So $n' = n$ and all $|x_i| > 0$, but this forces $B_{ij} = B'_{ij}$ $\forall i,j$ contradicting (W',S') a proper subgraph. ▨

REMARKS on the classification

(1) $\left\{\begin{array}{c}\text{symmetry groups } W \\ \text{of regular (convex)} \\ \text{polytopes}\end{array}\right\} = \left\{\begin{array}{l}\text{finite real ref'n} \\ \text{groups } W \text{ whose} \\ \text{Coxeter diagram} \\ \text{is a path}\end{array}\right\}$



(see EXERCISE 9 in Portugal Summer School list)

(2) Shephard & Todd (1955) assembled the classification of all finite complex ref'n groups G, first showing it comes down to those acting irreducibly on $V = \mathbb{C}^n$, and classifying those as the infinite family $G(r,p,n)$ $p \mid r$ plus 34 (!) exceptional cases.

(see Lehrer & Taylor's "Unitary ref'n groups" for a modern treatment.)

There are several notable subfamilies...

(3) Call a complex ref'n group G acting irreducibly on $V=\mathbb{C}^n$ well-generated if it can be generated by n complex ref'ns.

e.g.
- real ref'ns groups (W,S) with W finite
- $$G(r,1,n)=\left\langle \underset{s_1}{(1,2)}, \underset{s_2}{(2,3)}, \ldots, \underset{s_{n-1}}{(n-1,n)}, \begin{bmatrix} \zeta_r & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} \right\rangle$$ where $\zeta_r = e^{\frac{2\pi i}{r}}$
- $$G(r,r,n)=\left\langle \underset{s_1}{(1,2)}, \underset{s_2}{(2,3)}, \ldots, \underset{s_{n-1}}{(n-1,n)}, \left[\begin{array}{c|c} \begin{matrix} 0 & \zeta_r \\ \zeta_r^{-1} & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right] \right\rangle$$

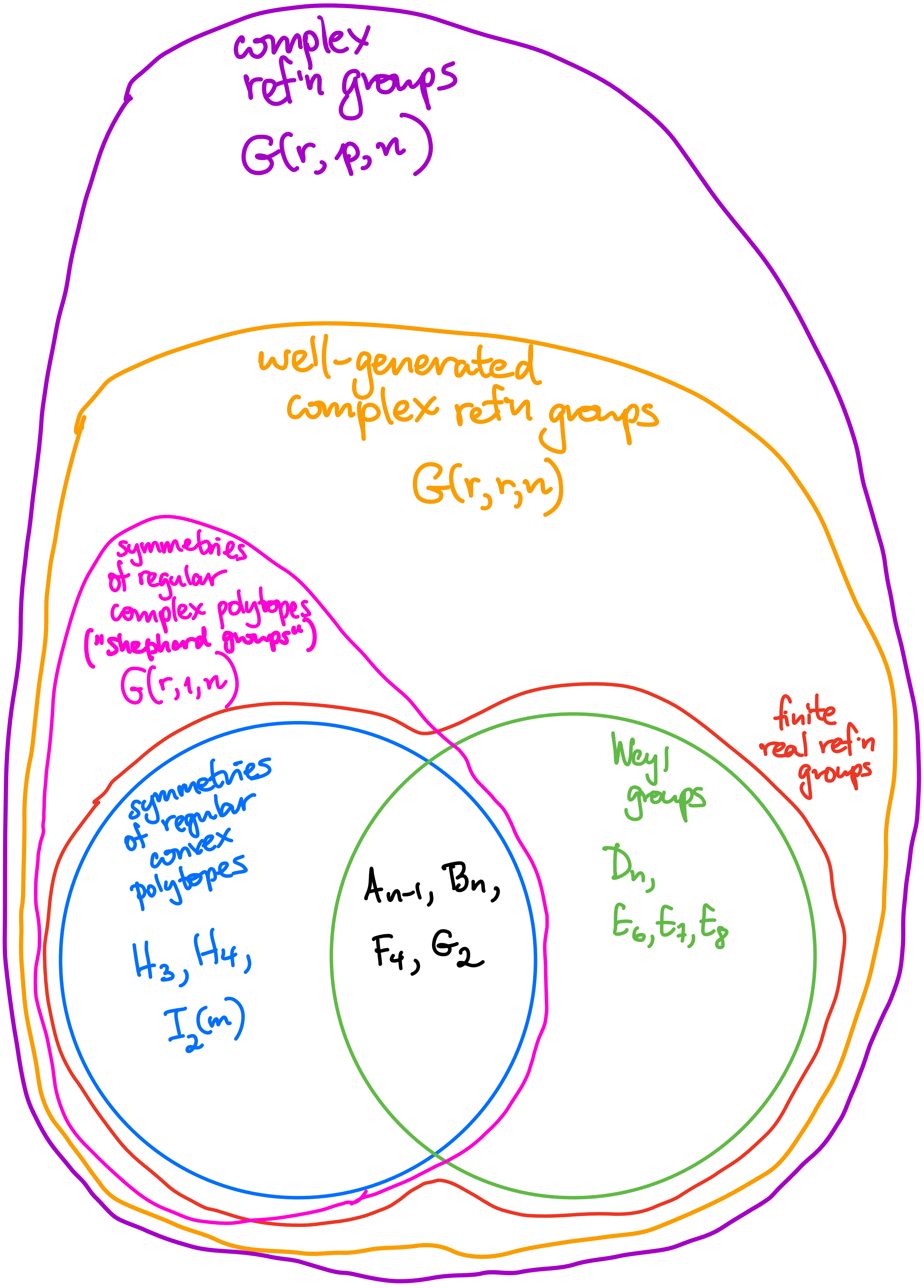
But some irreducible complex ref'n groups $G \subset GL_n(\mathbb{C})$ require $n+1$ ref'ns to generate

e.g. $$G(r,p,n)=\left\langle \underset{s_1}{(1,2)}, \underset{s_2}{(2,3)}, \ldots, \underset{s_{n-1}}{(n-1,n)}, \begin{bmatrix} \zeta_r^{\frac{r}{p}} & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}, \left[\begin{array}{c|c} \begin{matrix} 0 & \zeta_r \\ \zeta_r^{-1} & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right] \right\rangle$$

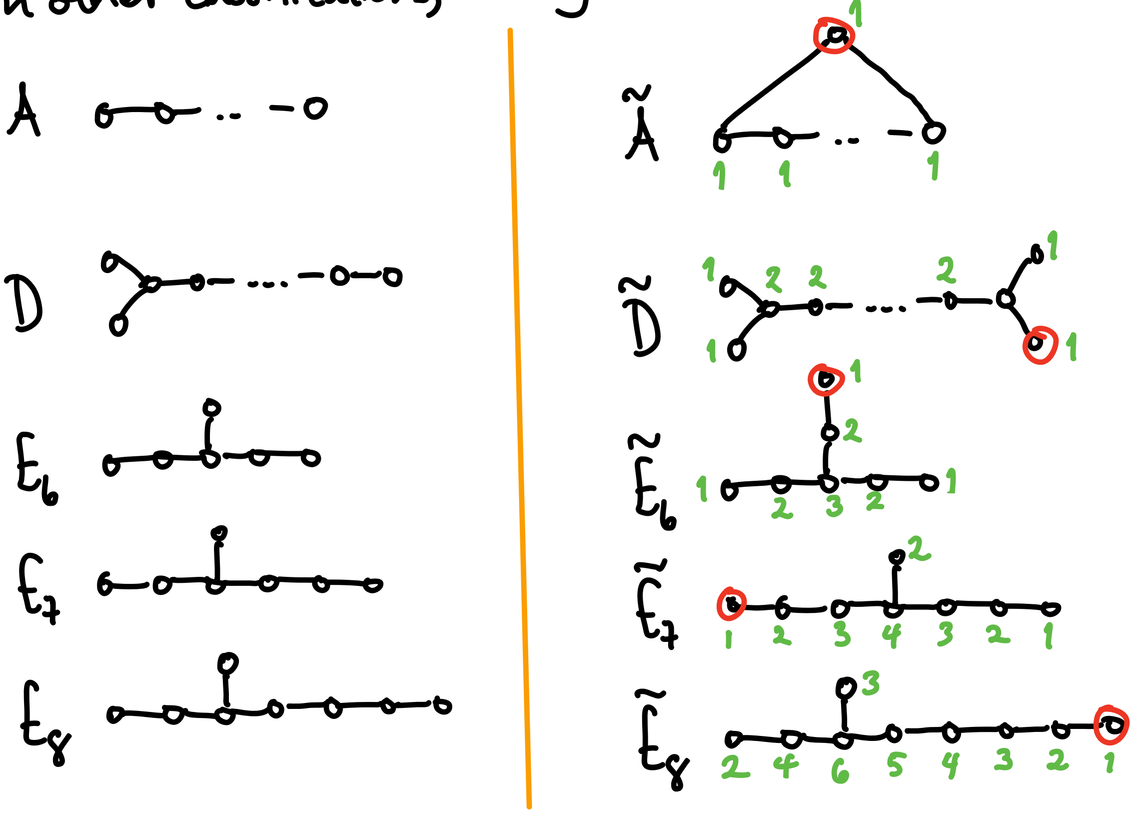
$n \geq 2, \; r \geq 2, \; \frac{r}{p} \geq 2$

(4) Even though convexity of polytopes makes little sense over $\mathbb{C}$, Shephard defined a good notion of regular complex polytopes, whose symmetry groups G are well-gen'd irreducible complex ref'n groups!

(see EXERCISE 9 in Portugal Summer School list again)


complex ref'n groups
$G(r,p,n)$
well-generated complex ref'n groups
$G(r,r,n)$
symmetries of regular complex polytopes ("shephard groups")
$G(r,1,n)$
finite real ref'n groups
symmetries of regular convex polytopes
$H_3, H_4, I_2(m)$
A_{n-1}, B_n, F_4, G_2
Weyl groups
D_n, E_6, E_7, E_8

(5) The simply-laced finite real ref'n groups are the A-D-E families, and occur unreasonably often in other classifications, along with their affines $\tilde{A}-\tilde{D}-\tilde{E}$:



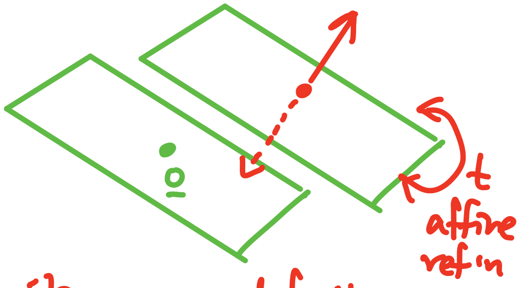
e.g. in

- singularity theory, as Kleinian singularities
- the finite subgroups of $SL_2(\mathbb{C})$, up to conjugation (McKay's correspondence)
- graphs with $\mathbb{R}_{>0}$-valued vertex-labeling f having
$$2f(v) = \sum_{\text{neighbors } v' \text{ of } v} f(v')$$
(f labeled on $\tilde{A}$-$\tilde{D}$-$\tilde{E}$ above)
- quivers with finitely many indecomposable rep'ns (Gabriel's Theorem)

(6) The affine Coxeter systems (W,S) all give rise to affine refl'n groups in $V=\mathbb{R}^n$ with pos. def. $(\cdot,\cdot)$

:= discrete groups generated by affine refl'ns

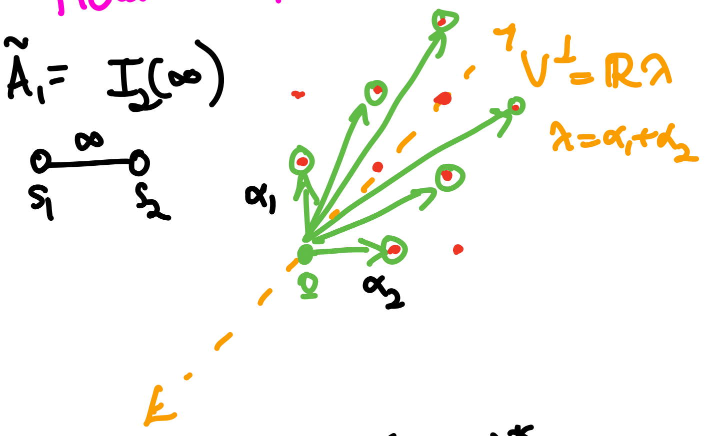
as explained in Humphreys Chap. 4 & §6.5 ...

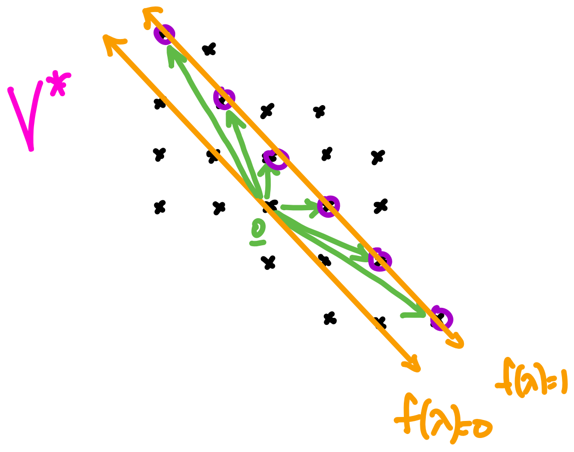


The bilinear form $B(\cdot,\cdot)$ on V is positive semidefinite with 1-dimensional $\ker B = V^{\perp} = \mathbb{R}\cdot\lambda$, so B induces a positive definite form on $V/V^{\perp}$.

Picture in V:

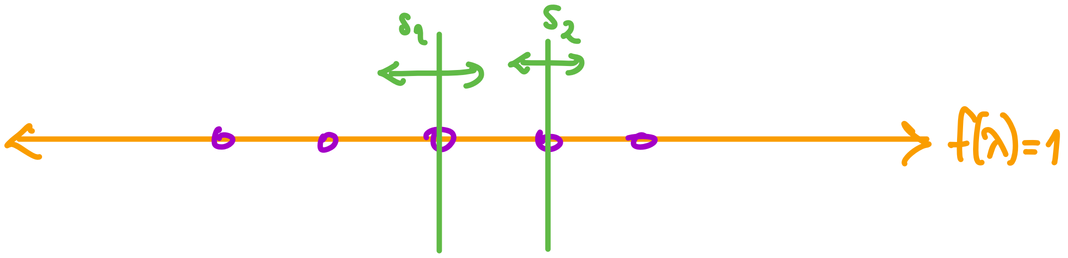
$\tilde{A}_1 = I_2(\infty)$





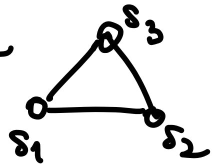
The dual space $(V/V^{\perp})^*$ is naturally the hyperplane $\{f\in V^*: f(\lambda)=0\}\subset V^*$ passing through 0, so it inherits a pos. definite inner product $(\cdot,\cdot)$ from $V/V^{\perp}$

Then one can translate this $(\cdot,\cdot)$ to the affine hyperplane $\{f\in V^*: f(\lambda)=1\}$, where W acts via affine reflections.



EXAMPLE:

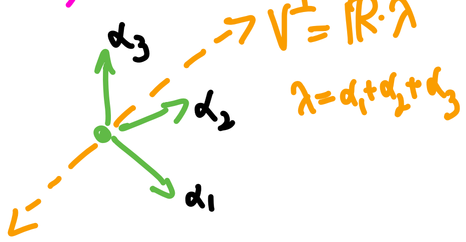
$\tilde{A}_2$



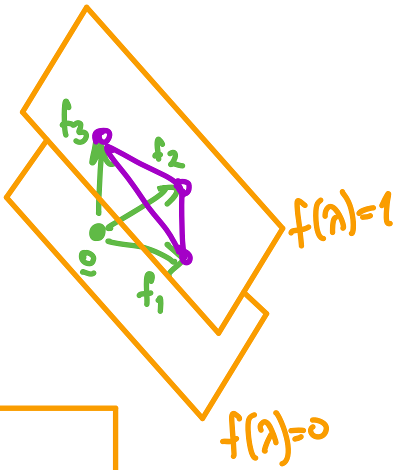
$$B(\cdot,\cdot)\quad \begin{array}{c|ccc} & \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ \alpha_2 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \alpha_3 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}$$

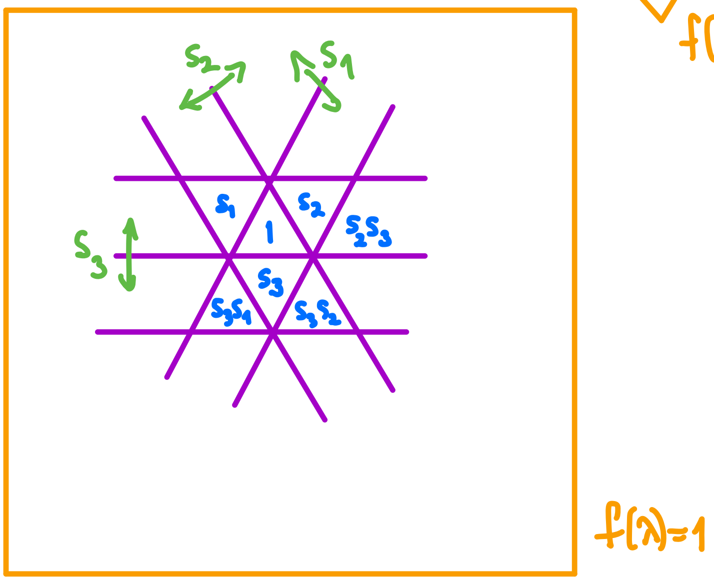
$\ker B = \mathbb{R}(\alpha_1+\alpha_2+\alpha_3)$

In V,



In V^*,





(7) The Weyl groups

(types $A_{n-1}, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$)

not only correspond to

- simple Lie algebras/groups,

and

- affine Coxeter systems $(\tilde{W}, \tilde{S})$

but also to

- cluster algebras of finite type (!)

(Fomin & Zelevinsky 2002)