PROP: (a) For any simplicial complex Δ,

$$\mathrm{Hilb}(K[\Delta]; t_1, \ldots, t_n) = \sum_{F \in \Delta} \prod_{i \in F} \frac{t_i}{1-t_i}$$

$$\left(\underset{\substack{\text{specialize} \\ t_i = t \ \forall i}}{\rightsquigarrow} \quad \mathrm{Hilb}(K[\Delta], t) = \sum_{F \in \Delta} \left(\frac{t}{1-t}\right)^{\#F} \right)$$

(b) For Δ partitionable as $\Delta = \bigsqcup_{i=1}^{s} [G_i, F_i]$,

$$\mathrm{Hilb}(K[\Delta]; t_1, \ldots, t_n) = \sum_{i=1}^{s} \frac{\prod_{j \in G_i} t_j}{\prod_{j \in F_i} (1-t_j)}$$

specialize $t_i = t$ $\rightsquigarrow$

$$\mathrm{Hilb}(K[\Delta], t) = \frac{\sum_{i=1}^{s} t^{\#G_i}}{(1-t)^d}$$

---

proof:

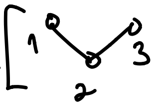
(a) $$\mathrm{Hilb}(K[\Delta]; t_1, \ldots, t_n) = \sum_{\underline{a} \in \mathbb{N}^n} \dim_K K[\Delta]_{\underline{a}} \cdot \underline{t}^{\underline{a}}$$

$$= \sum_{\substack{\underline{a} \in \mathbb{N}^n: \\ \mathrm{supp}(\underline{a}) \in \Delta}} \underline{t}^{\underline{a}} = \sum_{F \in \Delta} \sum_{\substack{\underline{a} \in \mathbb{N}^n: \\ \mathrm{supp}(\underline{a}) = F}} \underline{t}^{\underline{a}}$$

$$= \sum_{F\in\Delta} \prod_{i\in F} (t_i + t_i^2 + t_i^3 + \dots)$$

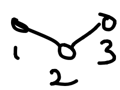
$$= \sum_{F\in\Delta} \prod_{i\in F} \frac{t_i}{1-t_i}$$

Feb 8, 2021

EXAMPLE: $\mathrm{Hilb}(K[$  $]; t_1, t_2, t_3)$

$$= 1 + \frac{t_1}{1-t_1} + \frac{t_2}{1-t_2} + \frac{t_3}{1-t_3} + \frac{t_1t_2}{(1-t_1)(1-t_2)} + \frac{t_2t_3}{(1-t_2)(1-t_3)}$$

$\big\downarrow\; t_1 = t_2 = t_3 = t$

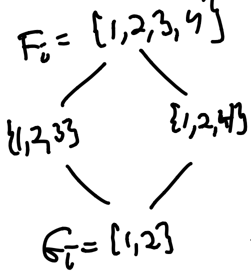
$\mathrm{Hilb}(K[$  $], t) = 1 + \frac{3t}{1-t} + 2\left(\frac{t}{1-t}\right)^2$

(6) If Δ is partitioned as $\Delta = \bigsqcup_{i=1}^{s} [G_i, F_i]$

$$\mathrm{Hilb}(K[\Delta]; \underline{t}) = \sum_{i=1}^{s} \sum_{\substack{\underline{a}\in\mathbb{N}^n:\\ \mathrm{supp}(\underline{a})\in[G_i,F_i]}} \underline{t}^{\underline{a}}$$

$$= \sum_{i=1}^{s} \prod_{j\in G_i} (t_j + t_j^2 + \dots) \cdot \prod_{j\in F_i\setminus G_i} (1 + t_j + t_j^2 + \dots)$$

EXAMPLE



$[G_i, F_i]$

$$\sum_{\substack{\underline{a}:\\ \mathrm{supp}(\underline{a})\in[12,1234]}} \underline{t}^{\underline{a}} = (t_1 + t_1^2 + \dots)(t_2 + t_2^2 + \dots)(1 + t_3 + t_3^2 + \dots)(1 + t_4 + t_4^2 + \dots)$$

$$= \sum_{i=1}^{s} \prod_{j \in G_i} \left(\frac{t_j}{1-t_j} \right) \prod_{j \in F_i \setminus G_i} \left(\frac{1}{1-t_j} \right)$$

$$= \sum_{i=1}^{s} \frac{\prod_{j \in G_i} t_j}{\prod_{j \in F_i} (1-t_j)}$$

$t_i = t \ \forall i$

$$\mathrm{Hilb}(k[\Delta], t) = \frac{\sum_{i=1}^{s} t^{\#G_i}}{(1-t)^d}$$

since $\#F_i = d$ $\forall i=1,\ldots,s$ as Δ is pure of dim $d-1$.

$\blacksquare$

A *shelling* of Δ is a stronger condition than partitioning, one with algebraic/topological consequences ...

DEF'N: For a pure $(d-1)$-dim'l simplicial complex Δ, a *shelling order* on its facets $F_1, F_2, \ldots, F_s$ is one such that $\forall i \geq 2$

$$\overline{F_i} \cap \left(\overline{F_1 \cup F_2 \cup \ldots \cup F_{i-1}} \right)$$

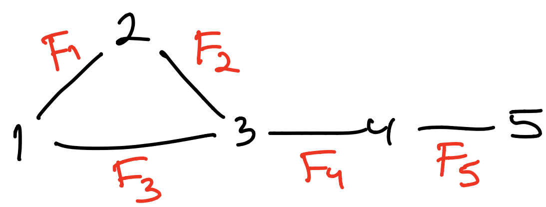
subcomplex gen'd by F_i, i.e. 2^{F_i}

is a *pure* $(d-2)$-dim'l subcomplex of $\overline{F_i}$.

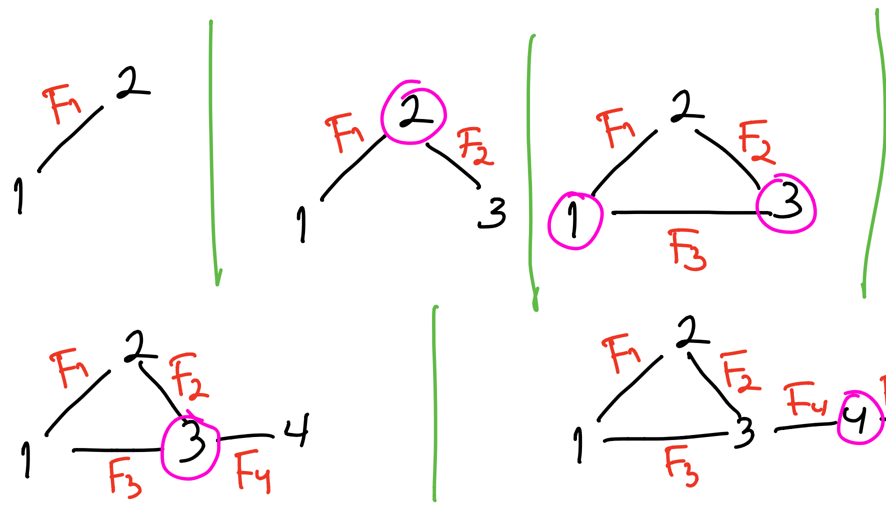
If such an order exists, say Δ is *shellable*.

EXAMPLES:

(1)



is a shelling order:



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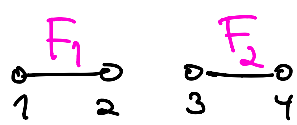
(2)

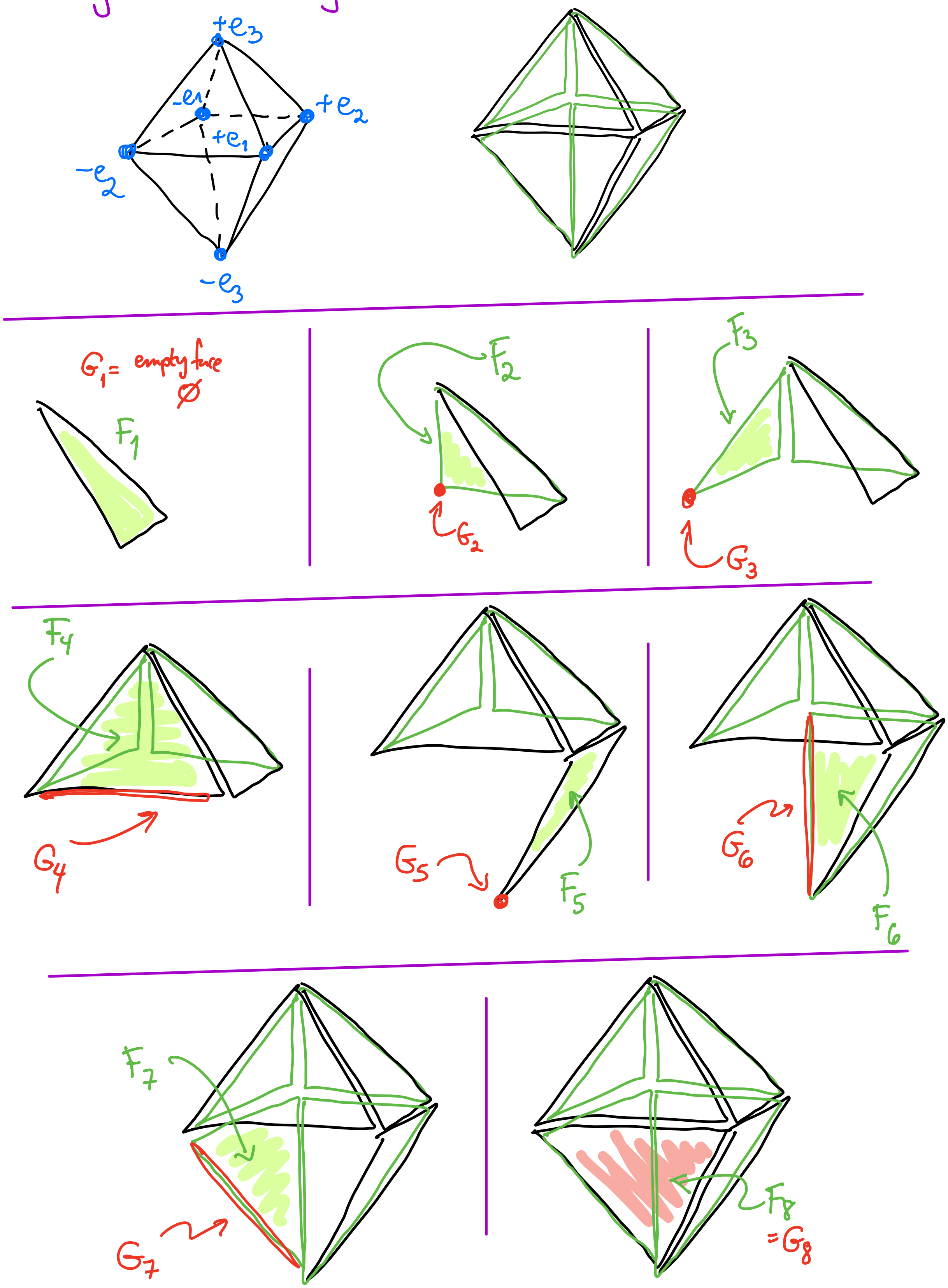


is not a shelling order

$$\overline{F_2} \cap \left(\overline{F_1}\right) = \{\emptyset\}$$

pure of dim -1, not 0

A shelling of the boundary of the octahedron

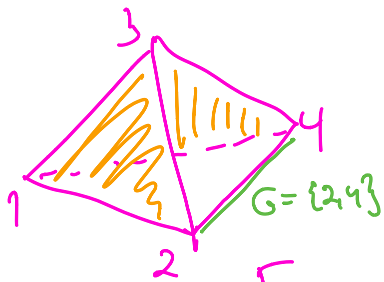


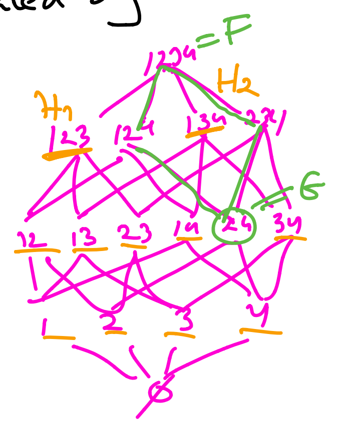
How does shelling relate to partitioning?

LEMMA: Pure $(d-2)$-dim'l subcomplexes of $(d-1)$-simplices $2^F = \overline{F}$ are the same as complements $2^F \setminus [G,F]$ for some face $G \subseteq F$.

Specifically, if the subcomplex is generated by $(d-2)$-faces $\underset{F-\{i_1\}}{\underset{\parallel}{H_1}}, \underset{F-\{i_2\}}{\underset{\parallel}{H_2}}, \ldots$

then $G = \{i_1, i_2, \ldots\}$





F $\quad$ $\underset{F-\{4\}=123}{\underset{\parallel}{H_1}}, \underset{134=F-\{2\}}{\underset{\parallel}{H_2}}$

proof: Proves itself! $\blacksquare$

PROPOSITION (Garsia 1980): A shelling order on Δ is a partitioning $\Delta = \bigsqcup_{i=1}^{s} [G_i, F_i]$ with an extra property:

$$G_i \notin F_1, F_2, \ldots, F_{i-1}.$$

proof: Given a shelling $F_1, F_2, \ldots, F_s$ of Δ

get a disjoint decomposition

$$\Delta = \bigsqcup_{i=1}^{s} \overline{F_i} \setminus (\underbrace{\overline{F_i} \cap (\overline{F_1} \cup \overline{F_2} \cup \ldots \cup \overline{F_{i-1}})}_{\text{pure } (d-2)\text{-dim'l, so of the form } 2^{F_i} \setminus [G_i, F_i]})$$

better notation: $\overline{\{F_1, F_2, \ldots, F_{i-1}\}}$

$$= \bigsqcup_{i=1}^{s} [G_i, F_i]$$

and the extra property $G_i \notin F_1, F_2, \ldots, F_{i-1}$ holds by construction.

Given a partitioning $\Delta \overset{(*)}{=} \bigsqcup_{i=1}^{s} [G_i, F_i]$ with the extra property, we claim $\forall i \geq 2$, $\overline{F_i} \cap (\overline{F_1 \cup \ldots \cup F_{i-1}}) = 2^{F_i} \setminus [G_i, F_i]$, which would show it's a shelling, by the LEMMA.

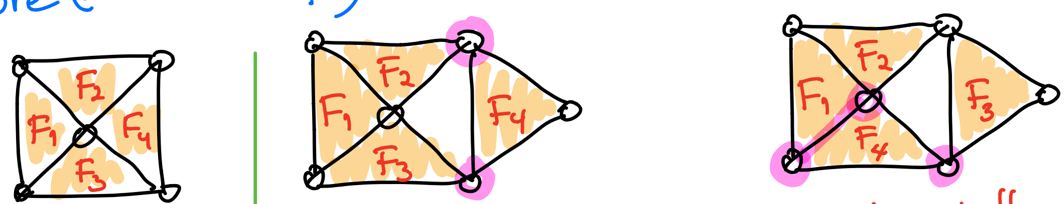
Check $\subseteq$ comes from disjointness in $(*)$

For $\supseteq$, note that any $H \in 2^{F_i} \setminus [G_i, F_i]$ has $H \subseteq F_i$ and $H \notin [G_j, F_j]$ for $j > i$, else $G_j \subseteq H \subseteq F_i$ violates the extra property.

Hence $H \in [G_j, F_j]$ for some $j = 1, 2, \ldots, i-1$, showing $H \in \overline{F_i} \cap (\overline{F_1 \cup \ldots \cup F_{i-1}})$, as desired.

$\blacksquare$

More (and better!) EXAMPLES of shellability, partitionability

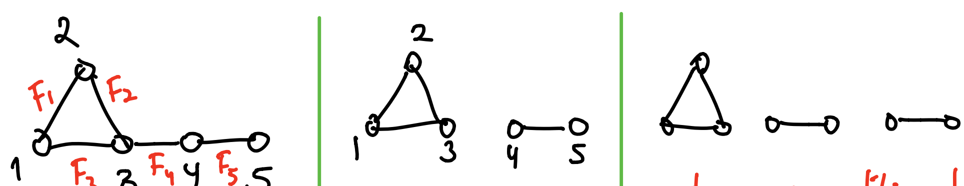


a shelling

not a shelling

not a shelling

(and this complex Δ is not shellable)



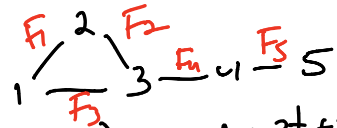
a shelling, so partitionable

Partitionable, not shellable!

not even partitionable, so not shellable

see new HW problem 9 from 1st half.

What does shellability give us?

EXAMPLE: $\Delta =$ 

$G_1 = \emptyset$
$G_2 = \{3\}$
$G_3 = F_3 = \{1,3\}$
$G_4 = \{4\}$
$F_5 = \{5\}$

has $\mathrm{Hilb}(K[\Delta], t) = \dfrac{1+3t+t^2}{(1-t)^2}$

We'll prove results implying, e.g....

- $K[\Delta]$ is finitely gen'd as a $K[\theta_1, \theta_2]$-module

where $\theta_1 = x_1 + x_3 + x_4$

$\theta_2 = x_2 + x_3 + x_5$

$$\left(\text{OR} \quad \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline \theta_1 & 1 & 0 & 1 & 1 & 0 \\ \theta_2 & 0 & 1 & 1 & 0 & 1 \end{array} \right)$$

regardless of field K

- $K[\Delta]$ is in fact a free $K[\underline{\theta}]$-module
$K[\theta_1, \theta_2]$

with basis elements $\{1, x_3, x_1x_3, x_4, x_5\}$
$\underline{x}^{G_1}$ $\underline{x}^{G_2}$ $\underline{x}^{G_2}$ $\underline{x}^{G_4}$ $\underline{x}^{G_5}$

Hence
$\mathrm{Hilb}(K[\Delta], t) = \mathrm{Hilb}(K[\underline{\theta}], t)\,(1 + t + t^2 + t + t)$

$$= \frac{1}{(1-t)^2}(1 + 3t + t^2)$$

DEF'N: R a ring, M a module means

a map $R \times M \to M$ with axioms ..

$(r, m) \mapsto rm$

For us, usually $R \underset{\text{subring}}{\subset} S$ and $M = S$

or M is an ideal in S

$$\left(\text{e.g. } \begin{array}{ccc} R & \subset & S \\ \shortparallel & & \shortparallel \\ k[\theta_1, \theta_2] & & K[\Delta] \\ & & \shortparallel \\ & & M \end{array}\right)$$

If $R = \bigoplus_{d=0}^{\infty} R_d$ is an $\mathbb{N}$-graded ring, then M is an $\mathbb{N}$-graded R-module if $M = \bigoplus_{d=0}^{\infty} M_d$ with

$$R_d \cdot M_e \subset M_{d+e}$$

Basic facts on when $M = \text{span}_R \{m_i\}_{i \in I}$

$$= \sum_i R m_i$$

M is spanned by $\{m_i\}_{i \in I}$ as R-mod

i.e. every $m \in M$ has a finite ~~expression~~ $m = \sum_{i=1}^{s} r_i m_i$, $r_i \in R$.

PROP: If $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ is a short exact sequence of R-mods,
(so $C \cong B/i(A)$)
then any R-mod generators $\{a_i\}_{i\in I}$ for A
$\{c_j\}_{j\in J}$ for C
give generators for B as $\{a_i\}_{i\in I} \cup \{b_j\}_{j\in J}$
where $\pi(b_j) = c_j \ \forall j \in J$
are any lifts of the c_j's

proof: Given $b \in B$, write

$$C \ni \pi(b) = \sum_j r_j c_j = \sum_j r_j \pi(b_j)$$

so $$\pi\left(b - \sum_j r_j b_j\right) = 0$$

$$b - \sum_j r_j b_j \in \ker(\pi)$$

$$\underbrace{b - \sum_j r_j b_j}_{= \sum_{i\in I} r_i' a_i} \in A$$

$$b = \sum_j r_j b_j + \sum r_i' a_i \quad \blacksquare$$

PROP: For an $\mathbb{N}$-graded ring $R = \bigoplus_{d=0}^{\infty} R_d$ and $\mathbb{N}$-graded R-module $M = \bigoplus_{d=0}^{\infty} M_d$

any homogeneous elements $\{m_i\}_{i\in I}$ have the property $\mathrm{span}_R \{m_i\}_{i\in I} = M$

$$\iff \mathrm{span}_{R_0} \{\overline{m_i}\}_{i\in I} = \underbrace{M/R_+M}_{\text{an } R/R_+ \,=\, R_0\text{-module}}$$

where $R_+ = R_1 \oplus R_2 \oplus \cdots$

EXAMPLE:

$$\mathrm{span}_{\underbrace{K[\theta_1,\theta_2]}_{R \,=\, \underbrace{R_0}_{=K} \oplus \underbrace{R_1 \oplus R_2 \oplus \cdots}_{R_+ = (\theta_1,\theta_2)_R}}} \{1, x_3, x_1x_3, x_4, x_5\} = \underbrace{K[\Delta]}_{M}$$

$$\iff \mathrm{span}_K \{\overline{1}, \overline{x_3}, \overline{x_1x_3}, \overline{x_4}, \overline{x_5}\} = \underbrace{K[\Delta]}_{M} \Big/ \underbrace{(\theta_1,\theta_2)_{K[\Delta]}}_{R_+M}$$

---

proof of PROP: $\mathrm{span}_R \{m_i\}_{i\in I} = M$

$$\Downarrow$$

$$\mathrm{span}_{\underbrace{R_0}_{R/R_+}} \{\overline{m_i}\}_{i\in I} = M/R_+M$$

For the reverse $\Leftarrow$, assume $\mathrm{span}_{R_0} \{\overline{m_i}\}_{i\in I} = M/R_+M$,

and show any homog. element $m \in M_d$ has $m \in \mathrm{span}_R \{m_i\}_{i\in I}$.

Use induction on d:

Write $\overline{m} = \sum_i r_i \overline{m}_i$ in M/R_+M

so $m = \sum_i r_i m_i + \underbrace{n}_{\text{in } R_+M}$ in M

Write
$n = \sum_j r_j' n_j$
with r_j' homog in R_+
n_j homog. in M

so $\underbrace{\deg(r_j')}_{>0} + \deg(n_j) = d \ \forall j$

$\Rightarrow \deg(n_j) < d$

so $n_j \in \operatorname{span}_R \{m_i\}_{i \in I}$

Hence $m = \sum_i r_i m_j + \sum_j r_j' n_j \in \operatorname{span}_R \{m_i\}_{i \in I}$ $\blacksquare$

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Recall $0 \to A \to B \to C \to 0$ s.e.s. of R-mods

R-gens $\{a_i\}_{i\in I}$ R-gens $\{c_j\}_{j\in J}$

lift $\{\tilde{c}_j\}_{j\in J}$

R-gens for B

$M = \bigoplus_{d=0}^{\infty} M_d$ a graded R-mod

homog. $\{m_i\}_{i\in I}$ have $\mathrm{span}_R\{m_i\} = M$

$$\Updownarrow$$

$\mathrm{span}_{R_0}\{\overline{m}_i\} = M/R_+M$

$R_0 = R/R_+$

Let's prove ...

<u>LEMMA</u>: $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_r)$ of degree one in $K[x_1, x_2, \ldots, x_n]$ for K a field with $\theta_i = \sum_{j=1}^{n} a_{ij} x_j$ $i = 1, \ldots, r$ $\left(\text{so } A = (a_{ij})_{\substack{i=1,\ldots,r \\ j=1,\ldots,n}}\right)$

has $K[\underline{x}]$ is a fin. gen'd $K[\underline{\theta}]$-module

$\iff K[\underline{x}] = K[\underline{\theta}]$ $\left(\text{so } K[\underline{x}] = \mathrm{span}_{K[\underline{\theta}]}\{1\}\right)$

$\iff \mathrm{span}_K\{\theta_1, \ldots, \theta_r\} = \mathrm{span}_K\{x_1, \ldots, x_n\}$

$\iff \mathrm{rank}_K(A) = n$ as a matrix.

proof: Think of

$$A = \begin{array}{c|c} & x_1 \; x_2 \cdots x_n \\ \begin{array}{c}\theta_1\\ \vdots \\ \theta_r\end{array} & \left[\begin{array}{ccc} & & \\ & a_{ij} & \\ & & \end{array}\right] \end{array}$$

use linear algebra to change this to

$$PAQ = s\left\{\left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline \underbrace{}_{s} \quad 0 & 0 \end{array}\right]\right.$$

with $P \in GL_r(K)$ row operations

$Q \in GL_n(K)$ column operations

This doesn't affect our question, renaming θ_i's via P,

and making a change-of-variables $\underline{y} = Q\underline{x}$ in our ring $K[x_1, \ldots, x_n]$ to $K[y_1, \ldots, y_n]$.

Now $\theta_1, \ldots, \theta_r$ is $y_1, \ldots, y_s$ $(\theta_i = y_i)$

and $K[\underline{\theta}] = K[y_1, \ldots, y_s]$ has $K[y_1, \ldots, y_s, \ldots, y_n]$ a fin. gen'd $K[\underline{\theta}]$-mod $\iff s = n$

since $K[y_1, \ldots, y_n]/(\underline{\theta})$

$= K[y_1, \ldots, y_n]/(y_1, \ldots, y_s)$

$\cong K[y_{s+1}, \ldots, y_n]$ is fin. gen'd$/K \iff s = n$. $\blacksquare$

THEOREM: Given $(\underline{\Theta}) = (\Theta_1, \ldots, \Theta_r) \in K[\Delta]_1$
(Stanley's green book Comb. & Comm. Alg. Lem. III.2.4)

with $\Theta_i = \sum_{j=1}^{n} a_{ij} x_j$ $\qquad$ $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$

and for any face $F \in \Delta$, let $\Theta_i\big|_F = \sum_{j \in F} a_{ij} x_j$

For K a field, T.F.A.E.

(a) $K[\Delta]$ is fin. gen'd as a $K[\underline{\Theta}]$

(b) $K[\Delta]/(\underline{\Theta})$ fin. gen'd over K (fin. dim'l K-vector space)

(c) $\forall$ faces $F \in \Delta$, $\quad K[x_j]_{j\in F}/(\underline{\Theta}|_F)$ is fin'd gen'd $/K$

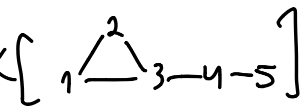
$\quad (\Leftrightarrow (c')\ K[x_j]_{j \in F} = K[\underline{\Theta}|_F]$

$\quad \Leftrightarrow (c'') \Leftrightarrow (c''')\ )$

(d) $\forall$ facets $F \in \Delta$, same thing as (c).

In any of these situations, $\{\underline{x}^F\}_{F\in\Delta}$ spans $K[\Delta]$ over $K[\underline{\Theta}]$

---

e.g. $K\Big[$  $\Big]$ has
$$\begin{array}{c} \\ \Theta_1 \\ \Theta_2 \end{array}\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}\end{array}$$

$\text{span}_{K[\underline{\Theta}]} \{\underline{x}^F\}_{F\in\Delta} = K[\Delta]$, regardless of K

proof: (a) $\Leftrightarrow$ (b) was our general lemma about spanning graded R-modules M.

For the remainder, note if $\Delta' \subset \Delta$ is a subcomplex, we get a ring surjection $K[\Delta] \xrightarrow{\pi} K[\Delta']$

Let's start thinking of $K[\Delta]$ as a $\underbrace{K[z_1,\ldots,z_r]}_{R}$-module

where z_i acts as mult. by θ_i, and then

z_i acts on $K[\Delta']$ by $\pi(\theta_i)$, etc.

This shows $(a) \Rightarrow (c) (\Rightarrow (d))$ since

$$K[\Delta] \overset{\pi}{\twoheadrightarrow} K[2^F] = K[x_j]_{j \in F} \quad \text{for any face } F$$

an R-module map

R has z_i acting as mult. by θ_i

R has z_i acting as mult. by $\theta_i|_F = \sum_{j \in F} a_{ij} x_j$

This also shows $(d) \Rightarrow (c)$, since every face is a subcomplex of a facet.

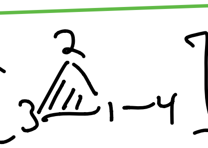
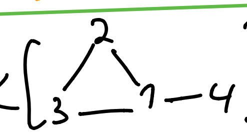
We only need to show $(d) \Rightarrow (a)$, and $K[\Delta] = \mathrm{span}_{K[\underline{\theta}]} \{\underline{x}^F\}_{F \in \Delta}$.

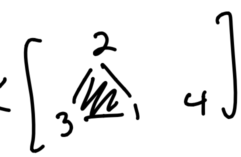
Let's prove this by induction on $\#\Delta$, using this s.e.s. for any facet F

$$0 \longrightarrow \underbrace{(\underline{x}^F)}_{\substack{\text{principal} \\ \text{ideal in } K[\Delta]}} \longrightarrow K[\Delta] \overset{\pi}{\longrightarrow} K[\Delta \setminus \{F\}] \longrightarrow 0$$

z_i acts as θ_i z_i acts as θ_i z_i acts as $\pi(\theta_i)$

$R = K[z_1,\ldots,z_r]$

e.g. $0 \to (x_1x_2x_3) \to K[$  $] \to K[$  $] \to 0$

$0 \to (x_1x_4) \to K[$  $] \to K[$  $] \to 0$

Note that, as R-modules, since $x_k \cdot \underline{x}^F = 0$ in $K[\Delta]$ for $k \notin F$,

$$\begin{array}{ccc}
K[x_j]_{j\in F} & \xrightarrow[\text{R-module isomorphism}]{\sim} & (\underline{x}^F) \\
1 & \longmapsto & \underline{x}^F \\
\prod_{j\in F} x_j^{a_j} & \longmapsto & \underline{x}^F \cdot \prod_{j\in F} x_j^{a_j}
\end{array}$$

R acts via z_i mult. by $\theta_i|_F$

R acts with z_i mults by θ_i

and gen'd over $K[\underline{\theta_i|_F}]$ by $\{1\}$ $\Longrightarrow$ $(\underline{x}^F)$ is gen'd as R-mod by $\{\underline{x}^F\}$

The s.e.s. now shows, via induction, that

$$K[\Delta] = \operatorname{span}_R\left(\{\underline{x}^F\} \cup \{\underline{x}^G\}_{G\in\Delta-\{F\}}\right)$$

$$= \operatorname{span}_R \{\underline{x}^G\}_{G\in\Delta} \quad \blacksquare$$

Math 8680 Feb. 15, 2021

THEOREM (Kind & Kleinschmidt 1979)
(Stanley's green book Chap III Thm. 2.5)

If Δ is a pure $(d-1)$-dim'l shellable complex
$\Delta = \bigsqcup_{i=1}^{s} [G_i, F_i]$ and $\underline{\Theta} = (\Theta_1, \ldots, \Theta_d)$ same d!
in $K[\Delta]_1$ with $K[\Delta]$ a fin. gen'd
$K[\underline{\Theta}]$-module,

then $\begin{cases} K[\underline{\Theta}] = K[\Theta_1, \ldots, \Theta_d] \text{ is a polynomial ring, i.e. } \underline{\Theta} \text{ are algeb. indep., and} \\ \boxed{K[\Delta] \text{ is a free } K[\underline{\Theta}]\text{-module, with basis } \{\underline{x}^{G_i}\}_{i=1,2,\ldots,s}} \end{cases}$

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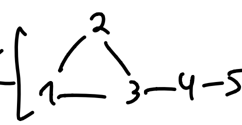
e.g. $(\Theta_1, \Theta_2) = \begin{pmatrix} x_1 + x_3 + x_4, \\ x_2 + x_3 + x_5 \end{pmatrix}$ inside $K\Big[$  $\Big]$

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proof: Note that the partitioning already implies

$$\mathrm{Hilb}(K[\Delta], t) = \frac{1}{(1-t)^d} \sum_{i=1}^{s} t^{\#G_i}$$

$$= \mathrm{Hilb}(M, t)$$

where M is a free R-module R^s of rank s
where $R = K[z_1, \ldots, z_d]$ and $M = R^s$
$\deg(z_i) = 1$ $\qquad = Re_1 \oplus \ldots \oplus Re_s$
with $\deg(e_i) = \#G_i$

Hence it suffices for us to show $\boxed{\mathrm{span}_{K[\underline{\theta}]}\left(\{\underline{x}^{G_i}\}_{i=1,\dots,s}\right) = K[\Delta]}$

since then we have a graded R-module surjection

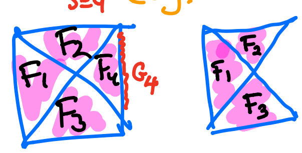
$$\begin{array}{rcl}
M & \twoheadrightarrow & K[\Delta] \\
\shortparallel & & \\
Re_1 \quad e_1 & \longmapsto & \underline{x}^{G_1} \ (=1 \text{ since } G_1=\emptyset) \\
\oplus & & \\
\vdots & & \\
\oplus & & \\
Re_s \quad e_s & \longmapsto & \underline{x}^{G_s} \\
\text{with } z_i & \longmapsto & \theta_i \text{ for } i=1,\dots,d
\end{array}$$

and $\mathrm{Hilb}(M,t) = \mathrm{Hilb}(K[\Delta],t)$

Let's show the boxed assertion by induction on $s = \#$ of shelling steps with this s.e.s. of $\underset{\substack{\shortparallel \\ K[z_1,\dots,z_d]}}{R}$-modules

$$0 \to (\underline{x}^{G_s}) \longrightarrow K[\Delta] \longrightarrow K[\underbrace{\Delta \setminus [G_s,F_s]}] \to 0$$

$s=4$ e.g.



Since F_s is the only facet containing G_s,

one has $x_j \cdot \underline{x}^{G_s} = 0$ unless $j \in F_s$.

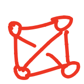
Hence one has an R-module iso.

$$\begin{array}{rcl}
K[x_j]_{j\in F_s} & \xrightarrow{\sim} & (\underline{x}^{G_s}) \\
1 & \longmapsto & \underline{x}^{G_s} \\
\prod_{j\in F_s} x_j^{a_j} & \longmapsto & \prod_{j\in F_s} x_j^{a_j} \cdot \underline{x}^{G_s}
\end{array}$$

where z_i acts as $\Theta_i|_{F_s}$ on the left
and acts as Θ_i on the right.

Hence $(\underline{x}^{G_s})$ is R-spanned by $\{\underline{x}^{G_s}\}$
and by induction using s.e.s,
$K[\Delta]$ is R-spanned by $\{\underline{x}^{G_i}\}_{i=1,\dots,s}$. ∎

DEF'N: Say $K[\Delta]$ is Cohen-Macaulay if
$\dim \Delta = d-1$ and one can extend K possibly
to find $(\underline{\Theta}) = (\Theta_1, \dots, \Theta_d) \in K[\Delta]_1$ with
$K[\Delta]$ is a free $K[\underline{\Theta}]$-module.

see HW EXERCISE 4(a) for why
K[]$_1$ has no such (Θ_1, Θ_2) unless $\#K \geq 3$

EXAMPLE:
Δ shellable
$\Rightarrow K[\Delta]$ C-M
$\forall$ fields K.

COROLLARY: $K[\Delta]$ is C-M with
$\dim \Delta = d-1$ and $f_0(\Delta) = n$

$$\Rightarrow \quad 0 \leq h_k[\Delta] \leq \underbrace{\binom{(n-d)+k-1}{k}}_{\substack{\#\text{ monomials of} \\ \text{deg } k \text{ in} \\ n-d \text{ variables}}} \quad \forall k$$

$$h_k[\Delta] = \dim_K \left(K[\Delta]/(\underline{\Theta}) \right)_k$$

proof: Since $K[\Delta]$ is C-M,

$K[\Delta] \cong K[\underline{\theta}]^s$ = free $K[\underline{\theta}]$-module on some homog. basis $\{b_1, \ldots, b_s\}$,

~~EXERCISE~~
M a fin. gen'd free R-mod graded $\Rightarrow$ $\exists$ a homog. R-basis

so one has

$$\frac{\sum_{k=0}^{d} h_k t^k}{(1-t)^d} = \mathrm{Hilb}(K[\Delta], t) = \mathrm{Hilb}(K[\underline{\theta}], t)\left(\sum_{i=1}^{s} t^{\deg(b_i)}\right)$$

$$\Rightarrow \begin{cases} \mathrm{Hilb}(K[\underline{\theta}], t) = \dfrac{1}{(1-t)^d} \\ \text{and } \sum_{k=0}^{d} h_k t^k = \sum_{i=1}^{s} t^{\deg(b_i)} \end{cases}$$

But then $K[\Delta]/(\underline{\theta}) \cong K^s$ with $\{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_s\}$ as K-basis

as graded K-vector spaces

so $\mathrm{Hilb}(K[\Delta]/(\underline{\theta}), t) = \sum_{i=1}^{s} t^{\deg(b_i)} = \sum_{k=0}^{d} h_k t^k$

i.e. $\dim_K \left(K[\Delta]/(\underline{\theta})\right)_k = h_k(\Delta)$.

$0 \leq$

Now note that if we pick any $(d-1)$-face F of Δ, then $\mathrm{span}_K \{\theta_i|_F\}_{i=1,\dots,d} = \mathrm{span}_K \{x_j\}_{j\in F}$

so
$$K[y_1,\dots,y_{n-d}] \twoheadrightarrow K[\Delta]/(\underline{\theta})$$
$\deg(y_i)=1$
$$y_i\text{'s} \longmapsto \{x_j\}_{j\notin F}$$

<u>surjects</u>

and hence $\dim\left(K[\Delta]/(\underline{\theta})\right)_k \leq \dim_K \left(K[\underline{y}]_k\right) = \binom{(n-d)+k-1}{k}$

$\blacksquare$