# Intersection Theory on Toric Varieties

4/30/2021

Based on paper by Fulton & Sturmfels '94

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Previously on Math 8680 ...

- Consider graded vector space
$$A^*(X) = \bigoplus_q A^q(X)$$
for some "geometric" object $X$.

- **Example:** Chow ring $A^*(M)_{\mathbb{R}}$ of a matroid $M$.

- **Problem:** prove $A^*(X)$ satisfies <u>PD</u>, <u>HL</u>, <u>HRM</u>

  $\Longrightarrow$ Towards log-concavity conjectures.

- **Approach:** reinterpret $A^*(M)$ as Chow ring of the Toric Variety $X(\Sigma_M)$ associated to the Bergman fan $\Sigma_M$

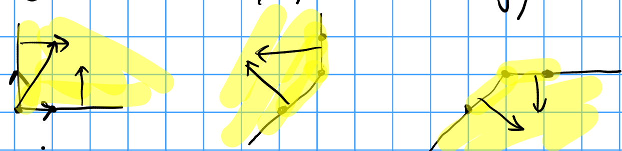
- **Lemma:** $A^*(X(\Sigma))$ enjoys a combinatorial interpretation as the ring of conewise continuous polynomials on $\Sigma$ modulo the ideal of globally linear functions.

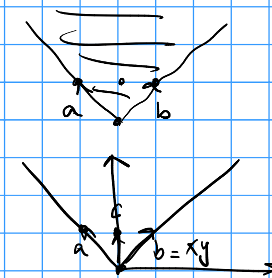
  $\hookrightarrow$ Fulton - Sturmfels '94
  Brion '96

# I. What is a toric variety? 

Ingredients:

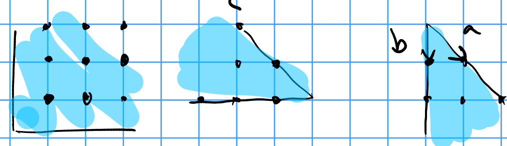
- $N=\mathbb{Z}^n$ lattice
- $M=\mathrm{Hom}(N,\mathbb{Z})$ dual lattice
- $\sigma\subset N$ rational polyhedral strongly convex <u>cone</u>






$x+y=2z$

- $\sigma^\vee\cap M=\{\mu\in M:\mu(v)\geq 0,\ \forall v\in\sigma\}\subset M$ <u>dual cone</u>




$\sigma^\vee\cap M\cong\mathbb{N}^2$

$\mathbb{C}[\mathbb{N}^2]\cong\mathbb{C}[a,b]$

$\mathbb{T}=(\mathbb{C}^*)^2$

- $\mathbb{C}[\sigma^\vee\cap M]$ a semigroup algebra

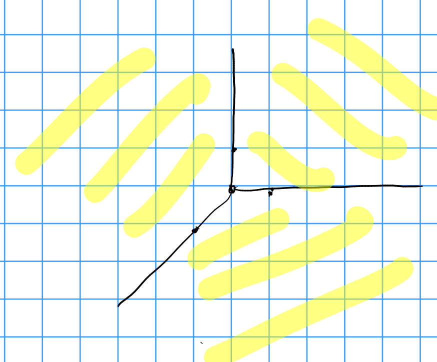
$$\cong\bigoplus_{v\in\sigma^\vee\cap M}\mathbb{C}x^v \qquad x^v\cdot x^\eta\subset x^{v+\eta}$$

$U_\sigma=\mathrm{Spec}\,\mathbb{C}[a,b]=\mathbb{A}^2_{\mathbb{C}}$

- $U_\sigma=\mathrm{Spec}\,\mathbb{C}[\sigma^\vee\cap M]$ a <u>toric affine variety</u>

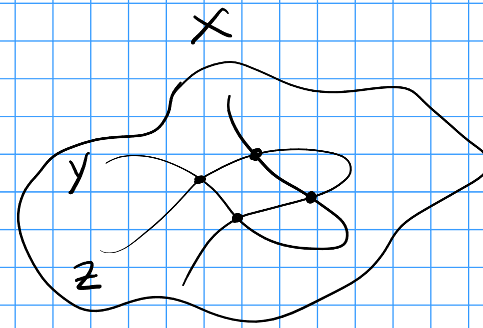
- $\Sigma$ a fan of cones

$\mathbb{P}^2$



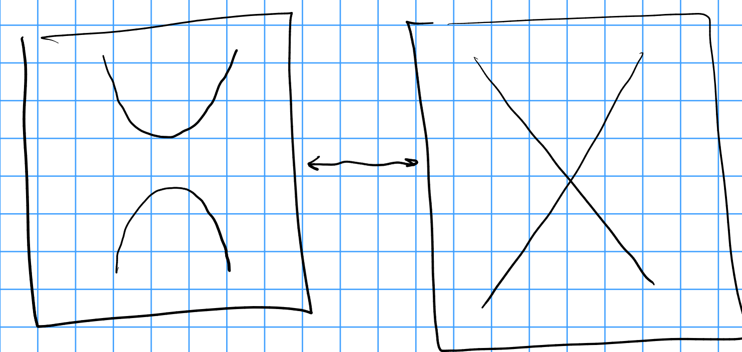
- Glue $U_\sigma$ for $\forall\sigma\in\Sigma$ to get a toric variety

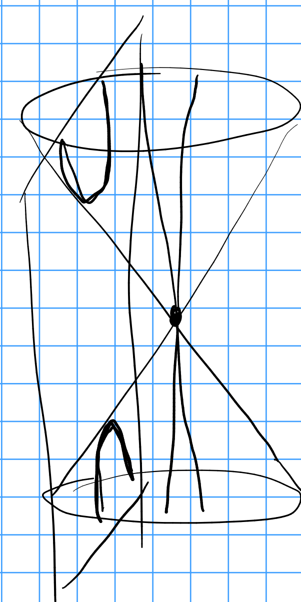
# II. What is intersection theory?




- Consider an algebraic variety $X$
- **Question:** How do subvarieties of $X$ intersect?
- **Approach:**
  - Consider the group of all subvarieties of $X$. $Z(X)$

    This group is graded by dimension. $Z(X) = \bigoplus Z_k(X)$
  - Quotient by rational equivalences





**Prop (Fulton–Sturmfels)** Let $X = X(\Sigma)$ be a toric variety.

- Chow group $A_k(X)$ is generated by the classes $[V(\sigma)]$ where $V(\sigma)$ determines a torus-invariant closed subvariety for all $\sigma \in \Sigma(k)$.
- the group of relations is generated by divisors $[\mathrm{div}(x^u)] = \sum_{\sigma} \langle u, n_{\sigma,\tau} \rangle \cdot [V(\sigma)]$ for $u$ a generator of $\sigma^\vee \cap M$ and $\tau \in \Sigma(k+1)$.

**Prop (Fulton–MacPherson–Sottile–Sturmfels)**

If $X$ is complete toric variety, $\underbrace{A^k(X)}_{} \longrightarrow \mathrm{Hom}(A_k(X), \mathbb{Z})$ is an isomorphism.

$\hookrightarrow$ maps from $A_i(Y) \to A_{i-k}(Y)$ $\forall Y \subset X$

## III. What is the Chow ring of a toric variety?

**Theorem / definition**

if $X$ is smooth & quasi-projective variety, then there is a unique product structure on $A(X)$ with:

- $Y, Z \subset X$ "generically transverse", then:

$$[Y]\cdot[Z] = [Y \cap Z]$$

This structure is the Chow ring:

$$A(X) = \bigoplus_{c=0}^{\dim X} A^c(X)$$

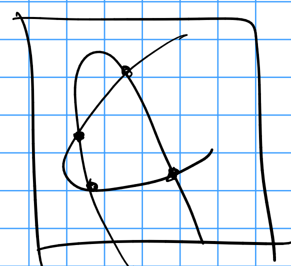
**Examples:**

- $A(\mathbb{A}^n) = \mathbb{Z}\cdot[\mathbb{A}^n] \cong \mathbb{Z}$

degree $d$ subvariety of codim $c$ will be $d \cdot x^c \in A^c(\mathbb{P}^n)$

- $A(\mathbb{P}^n) = \mathbb{Z}[x]/x^{n+1}$, where $x = [H] \in A^1(\mathbb{P}^n)$ is the class of a hyperplane $H \subset \mathbb{P}^n$

$\mathbb{P}^2$



$$2\cdot x^1 \cdot 2 \cdot x^1 = 4 \cdot x^2 = 4[P]$$

(*) Observe that an $(n-k)$-plane $L$ intersects a $k$-plane $M$ transversally at one point.

$$\Rightarrow \text{PD}: \quad A^{n-k}(X) \times A^k(X) \longrightarrow A^n(X) \cong \mathbb{Z}$$
$$[L]\cdot[M] \longmapsto \#\text{ of intersection points of } M \,\&\, L$$

- $A(\mathbb{P}^n \times \mathbb{P}^m) \cong A(\mathbb{P}^n) \otimes A(\mathbb{P}^m)$

(Künneth map)