METAPLECTIC ANALOGUES OF GELFAND–GRAEV MODELS

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1. Introduction

In this report, we study the representations of the Schur cover of the eleven (untwisted) finite groups of Lie type whose Schur multiplier is non-trivial. While the complex representations of finite groups of Lie type are well-known, it seems that there has been little research into their projective representations, or equivalently, the representations of the Schur covers of these groups. In most cases, the Schur multiplier $M(G)$ of a finite group of Lie type G is trivial, hence this reduces to the well-known theory. However, in small ranks and characteristics, the Schur multiplier of G can be non-trivial, and a list of all such groups is given in the following Theorem of Steinberg (1981). Following Steinberg's convention, we name these Chevalley groups by their corresponding Dynkin diagrams and specify the field $\mathbb{F}_q$ between the parentheses. A description of this classification can be found in [RH07].

Theorem 1.1. [Ste81, Theorem 1.1] Let G be a perfect simply connected Chevalley group over $\mathbb{F}_q$. Then the Schur multiplier of G is trivial except for the following cases:

(a) If G is $A_1(4)$, $A_2(2)$, $A_3(2)$, $C_3(2)$, $F_4(2)$ or $G_2(4)$, then $M(G) = \mathbb{Z}/2\mathbb{Z}$,
(b) If G is $A_1(9)$, $B_3(3)$ or $G_2(3)$, then $M(G) = \mathbb{Z}/3\mathbb{Z}$,
(c) If G is $A_2(4)$ then $M(G) = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$,
(d) If G is $D_4(2)$ then $M(G) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Note that each of these groups being perfect means that they have a universal central extension whose index is the size of the Schur multiplier.

Given a reductive group G, let its Schur multiplier be A. We then consider Schur covers $\widetilde{G}$ given by the short exact sequence

$$1 \to A \to \widetilde{G} \to G \to 1.$$

It is possible to construct representations of a reductive group G from representations of its subgroups; We set out to construct irreducible representations of $\widetilde{G}$ in an analogous way. In particular, we are interested in constructing *honest* representations—representations whose restriction to A does not just constitute the trivial representation. These are precisely the representations of $\widetilde{G}$ which cannot be obtained by inflating representations of G.

In the non-cover case, if B is a Borel subgroup of a reductive group G, with unipotent radical U and maximal torus T, we can write $B = U \rtimes T$. We then can take representations on T and inflate them

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to B by letting them act trivially on U. We can then induce these new representations to obtain either an irreducible representation of G, or a sum of two irreducibles (one of which is 1-dimensional). The interested reader is encouraged to consult [DL76] for more on constructing irreducible representations of G from those of the torus.

However, the same strategy does not work in the metaplectic case. Given a subgroup $H \leq G$ and the projection map $\widetilde{G} \xrightarrow{\phi} G$, let $\widetilde{H}$ denote the preimage $\phi^{-1}(H)$. Then Theorem 3.1 tells us that there exists $T' \leq \widetilde{T}$ such that $T' \cong T$ and

$$\widetilde{B} = \widetilde{U} \rtimes T'.$$

The analogous strategy would then be to inflate from T', but this would only give us the representations of $\widetilde{G}$ which arise via inflation from G and so does not account for the honest representations of $\widetilde{G}$. For this strategy to have yielded honest representations, we would have needed $\widetilde{B} = U' \rtimes \widetilde{T}$ instead, where $U' \cong U$ is a copy of U in $\widetilde{G}$, but Theorem 3.4 implies that there is no homomorphic section of U in $\widetilde{U}$.

Therefore, we turn to a different strategy for constructing irreducible representations of $\widetilde{G}$, again trying to emulate a strategy which worked for G. In the usual reductive group case, we have the following result of Gelfand and Graev, where U is the unipotent radical of a Borel subgroup B of a reductive group G:

Theorem 1.2. [Ste67, Theorem 49] Let G be a finite Chevalley group and let λ be a character on U such that $\lambda|_{X_a} \neq 1$ for simple roots a in the root system corresponding to G and $\lambda|_{X_a} = 1$ for a a positive but not simple root. Then $\mathrm{Ind}_U^G \lambda$ is multiplicity-free.

The induced representation $\mathrm{Ind}_U^G \lambda$ in the previous theorem is called the *Gelfand-Graev representation* of G. The goal of this project is to construct an analogue of the Gelfand-Graev representation in the metaplectic case for the 11 groups in Theorem 1.1. We have not however been able to find an exact analogue. Indeed, the data in Appendix B suggests that in general, a representation of the preimage $\phi^{-1}(U) = \widetilde{U}$ will not induce to be multiplicity-free in $\widetilde{G}$. However, we can still attempt to construct representations of $\widetilde{U}$ which induce to have minimal multiplicity in $\widetilde{G}$. In an attempt to better understand the representations of $\widetilde{U}$ and their relation to those of $\widetilde{G}$, we have the following results:

Theorem 1.3 (Corollary 3.10.1)**.** All 1-dimensional representations of $\widetilde{U}$ factor through U.

This result provides some information about the structure of the honest representations of $\widetilde{U}$. Furthermore, we have the following:

Theorem 1.4 (Theorem 3.16)**.** Let $1 \to A \to \widetilde{G} \xrightarrow{\phi} G \to 1$ be a central extension. Let H be a subgroup of G, and let $\widetilde{H} = \phi^{-1}(H)$ be the pre-image of H. Given representations π_1, π_2 of $\widetilde{H}$ such that

$$\left\langle Res_A^{\widetilde{H}} \pi_1, \mathrm{Res}_A^{\widetilde{H}} \pi_2 \right\rangle = 0$$

then the induced representations $\pi_1^{\widetilde{G}}$, $\pi_2^{\widetilde{G}}$ do not share any irreducible factors. In other words, we have:

$$\left\langle \chi_{\pi_1^{\widetilde{G}}}, \chi_{\pi_2^{\widetilde{G}}} \right\rangle = 0.$$

This theorem tells us that when we partition representations in the cover according to A-restrictions, induction preserves this distinction.

Our conjectures in §4 present several further avenues to explore irreducible representations of $\widetilde{U}$ and $\widetilde{G}$. Of particular interest is Conjecture 2, which claims that if two representations of $\widetilde{U}$ are isotypic components of the induced representation of the same character on A, then they will induce to the same representation on $\widetilde{G}$. This conjecture would provide a promising Gelfand–Graev analogue. Indeed, we would only need to select one representation in $\widetilde{U}$ for each character of A. The other conjectures primarily concern $\widetilde{U}$. Conjecture 4 posits that all honest representations of $\widetilde{U}$ have the same degree. Conjecture 3 claims that this degree is greater than or equal to the largest degree of an inflated representation of $\widetilde{U}$ from U. Conjecture 5 provides an elegant enumeration of the conjugacy classes of $\widetilde{U}$, which, if true, will be of great use for proving Conjectures 4 and 3.

2. Background

Our main object of study are the Schur covers of the 11 exceptional groups from Theorem 1.1, which are stem central extensions of these groups. In this section, we develop the requisite background on such extensions. We also introduce some background on Schur multipliers and Schur covers, as well as necessary terminology for representations and reductive groups.

Definition 2.1. A short exact sequence of groups

$$1 \to A \xrightarrow{i} G \to H \to 1$$

is a *central extension* of G by A if $i(A) \leqslant Z(G)$, where $Z(G)$ is the center of G. A central extension of G by A is *stem* if $A \leqslant G'$, where $G' = [G, G]$ is the derived subgroup of G.

Definition 2.2. Two central extensions of H by A,

$$1 \to A \to G \to H \to 1$$

and

$$1 \to A \to G' \to H \to 1,$$

are equivalent if there is a group isomorphism $\phi : G \to G'$ that commutes with the short exact sequences, i.e, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\ 1 & \longrightarrow & A & \longrightarrow & G' & \longrightarrow & H & \longrightarrow & 1. \end{array}$$

Definition 2.3. A central extension

$$1 \to A \to G \xrightarrow{\pi} H \to 1$$

is *split* if there exists a homomorphism $\phi : H \to G$ such that $\pi \circ \phi = \mathrm{id}_H$. In this case, $\phi(H) \cong H$ and G is a semi-direct product of A and $\phi(H)$.

We can often understand properties of the covering group through an understanding of cocycles. The relationship between these two objects is outlined below.

Proposition 2.4. [Wei69, Theorem 5.1.2] There is a bijection

$$\{\text{equivalence classes of central extensions of } G \text{ by } A\} \iff H^2(G, A).$$

Furthermore, the equivalence class of split extensions corresponds to the trivial cohomology class of $H^2(G, A)$ under this bijection.

The bijection is as follows: given a central extension

$$1 \to A \to \widetilde{G} \to G \to 1$$

take any (set-theoretic) section $\psi : G \to \widetilde{G}$. Then $\sigma : G \times G \to A$ given by $\sigma(g, h) = \psi(g)\psi(h)\psi(gh)^{-1}$ is a 2-cocycle, and thus $[\sigma] \in H^2(G, A)$.

Given $[\sigma] \in H^2(G, A)$, we construct a group $\widetilde{G} = \{(g, a) \mid g \in G, a \in A\}$ with multiplication given by $(g, a) \cdot (h, b) = (gh, \sigma(g, h)ab)$. The central extension is then

$$1 \to A \to \widetilde{G} \to G \to 1$$

where the middle maps are the natural ones.

The central extensions we study are those in which a group G is extended by its Schur multiplier, defined here along with notable facts about it.

Definition 2.5. The *Schur multiplier* of a group G is the group $M(G) = H_2(G, \mathbb{Z}) \simeq H^2(G, \mathbb{C}^\times)$.

Definition 2.6. Let G be a finite group. A *Schur covering group* (or *universal central extension*) of G is a central extension

$$1 \to M(G) \to \tilde{G} \to G \to 1$$

such that under the universal coefficients theorem map, defined as the map $H^2(G, M(G)) \to \mathrm{Hom}(M(G), M(G))$ in the short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(G^{ab}, M(G)) \to H^2(G, M(G)) \to \mathrm{Hom}(M(G), M(G)) \to 0,$$

the image of cohomology class corresponding to this extension is the identity map from $M(G)$ to itself.

Remark 2.7. In general, the Schur cover of a group is not universal, and G can have multiple isoclinic Schur covers. However, all of our groups G are perfect. This implies that G^{ab} is trivial, and thus that $\mathrm{Ext}^1_{\mathbb{Z}}(G^{ab}, M(G)) = 0$. Then we have an isomorphism $H^2(G, M(G)) \simeq \mathrm{Hom}(M(G), M(G))$, showing that the Schur cover is unique up to isomorphism.

Given a central extension, we often care whether or not it splits. One such criteria for splitting is given by the following:

Theorem 2.8 (Schur–Zassenhaus Theorem)**.** Let $1 \to A \to G \xrightarrow{\pi} H \to 1$ be a short exact sequence of groups. If $\gcd(|A|, |H|) = 1$, then the sequence splits.

One core property we make use of for representations of covers is whether or not they factor through a given surjection, defined as the following:

Definition 2.9. A representation $\pi : G \to GL(V)$ *factors* through a surjection $\phi : G \to U$ if $\phi(g) = \mathrm{Id}_V$ for all $g \in \ker \phi$. We call a representation π of $\widetilde{G}$ *honest* if it does not factor through the map $\widetilde{G} \to G$, and similarly for $\widetilde{U}$ and $\widetilde{B}$.

Finally, we define certain important subgroups of a reductive group G.

Definition 2.10. Let G be a reductive group over a field k. Let B be a fixed Borel subgroup of G and U be the unipotent radical of B. Let T denote the maximal split torus of B.

The subgroup U is of particular interest to us. It is this subgroup used in Theorem 1.2 to construct an elegant representation of G. For this reason, we would like to better understand U and $\widetilde{U}$ as we attempt to extend this theorem to the cover case.

3. Results

This section introduces our main results along with supporting background. Specifically, we describe our motivation and strategy for developing a Gelfand–Graev analogue for the metaplectic case.

We begin with a result showing how a naive attempt at parabolic induction fails in the cover case.

Theorem 3.1. Let G be a group in Theorem 1.1, and let $\widetilde{G}$ be the Schur cover of G, so that we have the short exact sequence

$$1 \to M(G) \to \widetilde{G} \xrightarrow{\pi} G \to 1$$

Let B, U, T be as in Definition 2.10. Let $\widetilde{B}$, $\widetilde{U}$, $\widetilde{T}$ be the preimages of B, U, T under $\pi : \widetilde{G} \to G$. We have the following:

(a) $1 \to M(G) \to \widetilde{T} \to T \to 1$ splits, so $T \simeq T'$ for some subgroup T' of $\widetilde{T}$,
(b) $\widetilde{B}$ is the normalizer of $\widetilde{U}$ in $\widetilde{G}$, and $\widetilde{B} = \widetilde{U} \rtimes T'$.

Proof. (a) Use Theorem 2.8, with the condition that $|T| = (q-1)^{\mathrm{rank}(G)}$. Note that $|M(G)|$ is always coprime to $|T|$ in these cases.
(b) Using Proposition 2.4, it is a straightforward computation to prove that $\widetilde{B} = \mathrm{N}_{\widetilde{G}}(\widetilde{U})$. Then use the fact that $\left|\widetilde{U}\right|$ is a power of q and Schur–Zassenhaus to get the desired conclusion. ∎

Because $\widetilde{B}$ splits over $T' \cong T$ rather than $\widetilde{T}$, we can only use the representations of T itself, which provides fewer than needed to do techniques in [DL76]. This motivates our desire for a Gelfand-Graev analogue.

Before embarking on our quest for such an analogue, we must introduce some definitions and results in group cohomology theory. For a full account of group cohomology, the reader is encouraged to consult [Wei69].

Definition 3.2. Let G be a group, M a G-module, and H a subgroup of G. We will make use of the following maps, defined for any $n \geqslant 0$. For full definitions, the reader is encouraged to read Section 2.3 of [Wei69].

(a) The *restriction* map $\mathrm{Res} : H^n(G, M) \to H^n(H, M)$.
(b) If $[G : H] < \infty$, the *corestriction* map $\mathrm{Cor} : H^n(H, M) \to H^n(G, M)$.
(c) If $H \trianglelefteq G$, the *inflation* map $\mathrm{Inf} : H^n(G/H, M^H) \to H^n(G, M)$, where $M^H = \{m \in M \mid hm = m$ for all $h \in H\}$.

Remark 3.3. If $[G : H] < \infty$, then by [Wei69, Corollary 2.4.9], the composition $\mathrm{Cor} \circ \mathrm{Res} : H^n(G, M) \to H^n(G, M)$ corresponds to multiplication by $[G : H]$.

In order to understand the representations of $\widetilde{U}$ and thus how to construct a Gelfand-Graev analogue, we have the following result.

Theorem 3.4. Let G be a group in Theorem 1.1. Using notation as in Theorem 3.1, we have that

$$1 \to A \to \widetilde{U} \to U \to 1 \tag{1}$$

is a non-split stem extension.

In order to prove Theorem 3.4, we develop several technical results.

Lemma 3.5. Let G be a group and M be a G-module. Assume that $H^2(G,M)$ is a p-group and that U is a finite index subgroup of G with $p \nmid [G:U]$. Then the restriction $H^2(G,M) \to H^2(U,M)$ is injective.

Proof. Consider the restriction and corestriction maps

$$H^2(G,M) \xrightarrow{\text{Res}} H^2(U,M) \xrightarrow{\text{Cor}} H^2(G,M)$$

which is the same as multiplying by $[G:U]$. Since $H^2(G,M)$ is a p-group and $p \nmid [G:U]$, $\text{Cor} \circ \text{Res}$ is an isomorphism. This implies Res is injective. □

Remark 3.6. The condition of Lemma 3.5 is satisfied if U is a finite index p-Sylow subgroup of G.

It is now useful to note that several of the groups relevant to our study are p-groups.

Lemma 3.7. [Ste67, Corollary to Theorem 26] For any finite Chevalley group G, U is a p-Sylow subgroup of G, where p is the characteristic of the field k.

By Theorem 1.1, we have that $M(G)$ is always a p-group. Since $|\widetilde{U}| = |M(G)| \cdot |U|$, it follows that $\widetilde{U}$ is a p-Sylow subgroup of $\widetilde{G}$ as well. Finally, we have the following p-groups.

Lemma 3.8. For G any of the groups in Theorem 1.1, both $A = H^2(G,\mathbb{C}^\times)$ and $H^2(G,A)$ are p-groups, where q is a power of p.

Proof. The first group is evidently a p-group from the list of Schur multipliers in Theorem 1.1, although a self-contained proof is also given in [Ste67, p. 47, Corollary 2]. For the second group, note that all such groups are perfect. Then Remark 2.7 completes the proof. □

We further require the following result:

Lemma 3.9. [Rot99, Lemma 7.64] For a central extension

$$1 \to A \to C \to G \to 1$$

we have that $A \leqslant C'$ if and only if the transgression map $\delta : H^1(A,\mathbb{C}^\times) \to H^2(G,\mathbb{C}^\times)$.

Finally, we note a necessary technical fact:

Lemma 3.10. [Wei69, Theorem 3.4.3] For a group G, a normal subgroup N of G, and a G-module A, we have a functorial short exact sequence (called the Inflation-Restriction exact sequence):

$$0 \to H^1(G/N, A^N) \xrightarrow{\text{Inf}} H^1(G,A) \xrightarrow{\text{Res}} H^1(N,A)^{G/N} \xrightarrow{\delta} H^2(G/N,A^N) \xrightarrow{\text{Inf}} H^2(G,A)$$

where $A^N = \{a \in A \mid na = a \text{ for all } n \in N\}$. The map $\delta : H^1(N,A)^{G/N} \to H^2(G/N,A^N)$ is called the *transgression map.*

Having stated the necessary results, we are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. The extension (1) corresponds to the cohomology class $\text{Res}([\sigma]) \in H^2(U,A)$, where $[\sigma] \in H^2(G,A)$ is the cohomology class corresponding to the Schur cover of G. In particular, Proposition 2.4 tells us $[\sigma]$ is non-trivial since $1 \to A \to \widetilde{G} \to G \to 1$ is a stem extension. Combining Lemma 3.5, 3.7, and 3.8, we have that $\text{Res} : H^2(G,A) \to H^2(U,A)$ is injective, hence $\text{Res}([\sigma])$ is also non-trivial, i.e: the extension (1) does not split.

On the other hand, applying Lemma 3.5, 3.7, and 3.8 again gives us an injective map

$$\text{Res} : H^2(G,\mathbb{C}^\times) \to H^2(U,\mathbb{C}^\times)$$

Using functoriality of the Inflation-Restriction exact sequence noted in Lemma 3.10, we have:

$$\begin{array}{ccc} H^1(A,\mathbb{C}^\times) & \xrightarrow{\sim} & H^2(G,\mathbb{C}^\times) \\ \big\downarrow{=} & & \big\downarrow{\text{Res}} \\ H^1(A,\mathbb{C}^\times) & \xrightarrow{\delta} & H^2(U,\mathbb{C}^\times) \end{array}$$

hence the transgression map is injective, thus Lemma 3.9 implies that (1) is a stem extension. □

This gives us the following corollary, which lets us better understand the honest and induced representations of $\widetilde{U}$.

Corollary 3.10.1. Applying the abelianization functor to (1) gives

$$A \to \widetilde{U}^{\text{ab}} \to U^{\text{ab}} \to 0$$

Theorem 3.4 implies that the first map is trivial, hence we have an isomorphism $\widetilde{U}^{\text{ab}} \simeq U^{\text{ab}}$. In particular, $[\widetilde{U} : \widetilde{U}'] = [U : U']$, and as a result all 1-dimensional representations of $\widetilde{U}$ factor through U.

Remark 3.11. The proof of Theorem 3.4 also applies for any subgroup H of G with index $[G : H]$ not divisible by p. In particular, all 1-dimensional representations of $\widetilde{B}$ and $\widetilde{G}$ also factor through B and G, respectively.

We now develop some technical results describing the relationship between representations of A and of the pullback $\widetilde{H}$ of some subgroup $H \subset G$. These will be used to prove a structure theorem connecting inductions of representations in $\operatorname{Irr} \widetilde{H}$ to their restrictions to A.

Proposition 3.12. [GR18, Exercise 4.11] For any (finite-dimensional) representations $\pi : H \to GL(V)$ with induced representation π^G, let R be a set of right coset representatives of G over H. Then we have:

$$\operatorname{tr}(\pi^G(g)) = \sum_{r \in R} \begin{cases} \operatorname{tr}(\pi(h)) & h = rgr^{-1} \in H \\ 0 & \text{otherwise} \end{cases}$$

Proof. Define f_{r,e_i} for $r \in G$, e_i a standard basis vector of V as

$$f_{r,e_i}(x) = \begin{cases} \pi(h) \cdot e_i & x = hr \text{ for } h \in H \\ 0 & \text{otherwise} \end{cases}$$

Then $\{f_{r,e_i} \mid r \in R\}$ is a basis of the vector space V^G. Notice for fixed i

$$\left(\pi^G(g) f_{r,e_i}\right)(x) = f_{rg^{-1},e_i}(x)$$

Thus $\left(\pi^G(g) f_{r,e_i}\right)(x)$ is in the span of $\{f_{rg^{-1},e_j} \mid 1 \le j \le dim(V)\}$ since these are the only basis functions which are nonzero on the right coset Hrg^{-1}, and they are zero on the rest of G. If $Hrg^{-1} \neq Hr$, then in a matrix representation of $\pi^G(g)$ with basis $\{f_{r,e_i} \mid r \in R\}$, the entry in the f_{r,e_i}, f_{r,e_i} cell is 0. If $Hrg^{-1} = Hr$, then plugging in $x = rg^{-1}$, we find

$$\left(\pi^G(g) f_{r,e_i}\right)(x) = \sum_j \left\langle \pi(rgr^{-1}) e_i, e_j \right\rangle f_{r,e_j}$$

Summing over all basis vectors and cosets yields the proposition. □

The following result displays the induction of the trivial character on A to any subgroup $\widetilde{H}$ containing A.

Proposition 3.13. For $\chi \in \operatorname{Irr}(A)$, for all $(H, a) \in \widetilde{H}$ where $H \subset G$,

$$(\operatorname{Ind}_A^{\widetilde{H}} \chi)((h, a)) = \begin{cases} |H| \chi(a) & \text{if } h = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Denote by $\widetilde{H} \setminus A$ the set of left coset representatives of A in $\widetilde{H}$. Then

$$(\operatorname{Ind}_A^{\widetilde{H}} \chi)((h, a)) = \sum_{\tilde{h} \in \widetilde{H} \setminus A} \chi(\tilde{h}(h, a)\tilde{h}^{-1})$$

where we understand $\chi(\tilde{h}(h, a)\tilde{h}^{-1}) = 0$ if $\tilde{h}(h, a)\tilde{h}^{-1} \notin A$. But since A is in the center of $\widetilde{H}$, $\tilde{h}(h, a)\tilde{h}^{-1} \in A$ if and only if $(h, a) \in A$, i.e. $h = 1$, in which case $\tilde{h}(1, a)\tilde{h}^{-1} = (1, a)$. The proposition then follows from $[\widetilde{H} : A] = |H|$. □

Furthermore, we can quickly show that distinct characters on A induce to have disjoint support on $\widetilde{H}$.

Proposition 3.14. For $\chi_1, \chi_2 \in \operatorname{Irr}(A)$,

$$\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi_1, \operatorname{Ind}_A^{\widetilde{H}} \chi_2 \right\rangle_{\widetilde{H}} = 0$$

Proof. By Frobenius reciprocity,

$$\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi_1, \operatorname{Ind}_A^{\widetilde{H}} \chi_2 \right\rangle_{\widetilde{H}} = \left\langle \chi_1, \operatorname{Res}_A^{\widetilde{H}} \operatorname{Ind}_A^{\widetilde{H}} \chi_2 \right\rangle_A$$

But by Proposition 3.13, $\operatorname{Res}_A^{\widetilde{H}} \operatorname{Ind}_A^{\widetilde{H}} \chi_2 = |H| \cdot \chi_2$. The claim then follows from $\langle \chi_1, \chi_2 \rangle_A = 0$. □

Finally, we combine the previous two results to get an elegant characterization of the restriction of representations $\widetilde{H}$ to A, or equivalently of the induction of any character of A to $\widetilde{H}$.

Proposition 3.15. For all $\chi \in \operatorname{Irr} A$ and $\pi \in \operatorname{Irr} \widetilde{H}$,

$$\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi, \pi \right\rangle_{\widetilde{H}} = \begin{cases} \dim \pi & \text{if } \left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi, \pi \right\rangle_{\widetilde{H}} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. By Frobenius reciprocity,

$$\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi, \pi \right\rangle_{\widetilde{H}} = \left\langle \chi, \operatorname{Res}_A^{\widetilde{H}} \pi \right\rangle_A .$$

But Proposition 3.14 implies that there is a unique χ such that the inner product on the left hand side is non-zero. In particular, if $\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi, \pi \right\rangle_{\widetilde{H}} \neq 0$, then $\operatorname{Res}_A^{\widetilde{H}} \pi = k \cdot \chi$ for some positive integer k. Hence $k = k \cdot \chi(1) = \operatorname{Res}_A^{\widetilde{H}} \pi(1) = \pi(1) = \dim \pi$. It follows that

$$\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi, \pi \right\rangle_{\widetilde{H}} = \langle \chi, k \cdot \chi \rangle_A = k = \dim \pi.$$

□

In light of Propositions 3.13 and 3.15, we see that each irreducible representation π of a subgroup $\widetilde{H} \subset \widetilde{G}$ corresponding to $H \subset G$ restricts to a single representation χ on A. This sets up the following theorem, which tells us that representations $\pi \in \operatorname{Irr} \widetilde{H}$ restricting to different $\chi \in \operatorname{Irr} A$ will then induce to orthogonal representations on $\widetilde{G}$.

Theorem 3.16. Let $1 \to A \to \widetilde{G} \xrightarrow{\phi} G \to 1$ be a central extension. Let H be a subgroup of G, and let $\widetilde{H} = \phi^{-1}(H)$ be the pre-image of H. Given irreducible representations π_1, π_2 of $\widetilde{H}$, we have $\operatorname{Res}_A^{\widetilde{H}} \pi_i = \dim \pi_i \cdot \chi_i$ for irreducible representations χ_i of A. Then if $\chi_1 \neq \chi_2$, the induced representations $\pi_1^{\widetilde{G}}$ and $\pi_2^{\widetilde{G}}$ do not share any irreducible components. In other words, we have:

$$\left\langle \chi_{\pi_1^{\widetilde{G}}}, \chi_{\pi_2^{\widetilde{G}}} \right\rangle = 0$$

Proof of Theorem 3.16. We note that $\pi_i \subset \operatorname{Ind}_A^{\widetilde{H}} \chi_i$ and thus that $\operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \pi_i \subset \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \operatorname{Ind}_A^{\widetilde{H}} \chi_i = \operatorname{Ind}_A^{\widetilde{G}} \chi_i$ In particular, this means that $\operatorname{Res}_A^{\widetilde{G}} \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \pi_i = [\widetilde{G} : A]\chi_i$. Then if $\chi_1 \neq \chi_2$, we have that

$$\left\langle \operatorname{Res}_A^{\widetilde{G}} \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \pi_1, \operatorname{Res}_A^{\widetilde{G}} \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \pi_2 \right\rangle = \left\langle [\widetilde{G} : A]\chi_1, [\widetilde{G} : A]\chi_2 \right\rangle = 0.$$

It follows that $\left\langle \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \pi_1, \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}} \pi_2 \right\rangle = 0$, as desired. □

Theorem 3.16 brings us closer to developing a Gelfand–Graev analogue by giving us a way of partitioning representations of $\widetilde{H}$ which commutes with induction. Namely, representations are partitioned according to their restrictions on A. However, it fails to provide a strict enough classification to give a useful analogue.

4. Conjectures and Motivating Results

This section attempts to chart potential directions for future work on this topic. Most importantly, Conjecture 2 provides a converse to Theorem 3.16 which would provide an elegant Gelfand–Graev analogue. In addition to being motivated by the theorem, this conjecture is a stronger version of Conjecture 1. Later conjectures present predictions on the structure of the honest representations of $\widetilde{U}$ and of the conjugacy classes of $\widetilde{U}$. Such information would be valuable in further understanding honest representations of $\widetilde{U}$ and, through induction, of $\widetilde{G}$.

The data of $SL(3, 2)$ and $SL(2, 9)$, as described in Appendix B, suggest the following conjecture:

Conjecture 1. Let $\mu, \mu' \in \operatorname{Irr}(\widetilde{U})$ are honest characters of dimension > 1. Then $\operatorname{Ind}_{\widetilde{U}}^{\widetilde{G}} \mu \cong \operatorname{Ind}_{\widetilde{U}}^{\widetilde{G}} \mu'$ if and only if $\mu \otimes \chi = \mu'$ for an irreducible 1-dimensional character χ of $\widetilde{U}$.

The strongest definite knowledge related to this conjecture is Proposition 4.3. We first prove a lemma about the center of $\widetilde{U}$.

Lemma 4.1. $Z(\widetilde{U}) = A$, where A refers to the embedding of A in $\widetilde{U}$.

Proof. Choose w_o to be the longest word in the Weyl group W. Then for any $w \in W$ and root α, we have that $wU_\alpha w^{-1} = U_{w\alpha}$. Then since w_o is the unique element which sends every positive root to a negative root, $w_o U w_o^{-1} = U^-$, the opposite unipotent subgroup. Then $w_o U w_o^{-1} \cap U = 1$. By the cocycle presentation, it is obvious that $w_o \widetilde{U} w_o^{-1} \cap \widetilde{U} \subseteq \{(1,a) : a \in A\} = A$, where by A we mean the embedding of A in $\widetilde{U}$. So $Z(\widetilde{U}) \subseteq w_o \widetilde{U} w_o^{-1} \cap \widetilde{U} \subseteq A$. On the other hand, by definition of central extensions, $A \subseteq Z(\widetilde{U})$. The desired equality follows. □

We now state Mackey's restriction formula, which will be useful in proving our next result.

Theorem 4.2. [CR87, Theorem 10.13] Let π be a representation of $H \subset G$, $H_g = H \cap gHg^{-1}$, and $\pi_g(x)(v) = \pi(g^{-1}xg)v$. Then

$$\operatorname{Res}_H^G \operatorname{Ind}_H^G \pi \cong \bigoplus_{H\backslash G/H} \operatorname{Ind}_{H_g}^H \pi_g.$$

Now we can show the following proposition, showing that two representations of $\widetilde{U}$ which are "tensor-equivalent" induce to share at least some components in $\widetilde{G}$.

Proposition 4.3. Suppose $\pi, \chi \in \operatorname{Irr}(\widetilde{U})$ and χ is 1-dimensional. Then

$$\left\langle (\pi \otimes \chi)^{\widetilde{G}}, \pi^{\widetilde{G}} \right\rangle_{\widetilde{G}} > 0.$$

Proof. By Frobenius reciprocity, this statement is equivalent to $\left\langle \operatorname{Res}_{\widetilde{U}}^{\widetilde{G}} (\pi \otimes \chi)^{\widetilde{G}}, \pi \right\rangle_{\widetilde{U}} > 0$. By Mackey's restriction formula Theorem 4.2,

$$\begin{aligned} &\left\langle \operatorname{Res}_{\widetilde{U}}^{\widetilde{G}} (\pi \otimes \chi)^{\widetilde{G}}, \pi \right\rangle_{\widetilde{U}} \\ &= \left\langle \bigoplus_{g \in \widetilde{U}\backslash\widetilde{G}/\widetilde{U}} \operatorname{Ind}_{\widetilde{U}_g}^{\widetilde{U}} (\pi \otimes \chi)_{\widetilde{U}_g}, \pi \right\rangle_{\widetilde{U}_w} \\ &= \sum_{tw \in TW} \left\langle \operatorname{Ind}_{\widetilde{U}_{tw}}^{\widetilde{U}} (\pi \otimes \chi)_{\widetilde{U}_{tw}}, \pi \right\rangle \\ &= \sum_{tw \in TW} \left\langle (\pi \otimes \chi)_{\widetilde{U}_{tw}}, \operatorname{Res}_{\widetilde{U}_{tw}}^{\widetilde{U}} \pi \right\rangle \end{aligned}$$

The desired inequality is then satisfied if for some $tw \in TW$, $\operatorname{Res}_{\widetilde{U}_{tw}}^{\widetilde{U}} \chi$ is trivial. We showed in the proof of Lemma 4.1 that for $w_o \in W$ the longest word in the Weyl group, $\widetilde{U}_{w_o} = A$. From Corollary 3.10.1, we know that χ is trivial on A. Hence letting $tw = 1 \cdot w_o$ implies the result. □

However, it is likely that the relevant property of honest representations in $\operatorname{Irr}(\widetilde{U})$ is not tensor-equivalence, but rather which irreducible representation of A they restrict to. Indeed, any 1-dimensional representation χ of $\widetilde{U}$ is trivial on A by Corollary 3.10.1. Therefore, for $\pi, \chi \in \operatorname{Irr}(\widetilde{U})$ where χ is 1-dimensional, $\operatorname{Res}_A^{\widetilde{U}} \pi = \operatorname{Res}_A^{\widetilde{U}} \pi \otimes \chi$. Further, we have that for any $\pi \in \operatorname{Irr}(\widetilde{H})$, there always exists $\chi \in \operatorname{Irr}(A)$ such that $\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi, \pi \right\rangle \neq 0$ by Frobenius reciprocity. Moreover, Proposition 3.14 says that there is a unique such χ.

We have the following as a final demonstration of how nicely the representations on A relate to those on higher subgroups before stating the conjecture.

Proposition 4.4. Let $\{\pi_i\} \subset \operatorname{Irr}(\widetilde{H})$ be the set of irreducible representations of $\widetilde{H}$ which factor through $H \subset G$, and let χ_{tr} be the trivial representation on A. Then

$$\operatorname{Ind}_A^{\widetilde{H}} \chi_{tr} = \bigoplus_i (\pi_i)^{\oplus \dim \pi_i}.$$

Proof. Suppose that $\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi_{tr}, \pi \right\rangle_{\widetilde{H}} > 0$ for some $\pi \in \operatorname{Irr}(\widetilde{H})$. Then by Proposition 3.13, we see that $\operatorname{Res}_A^{\widetilde{H}} \pi(a) = \mathrm{Id}_{\dim \pi}$. Therefore, $\pi \in \{\pi_i\}$.

Now suppose that we have some π which factors through H. Then we have that

$$\left\langle \operatorname{Ind}_A^{\widetilde{H}} \chi_{tr}, \pi \right\rangle = \frac{1}{|G|} \sum_{g \in G} \operatorname{Ind}_A^{\widetilde{H}} \chi_{tr}(g) \overline{\pi(g)} = \frac{1}{|G|} \sum_{g \in G} \left(\sum_{r \in R} \chi_{tr}(rgr^{-1}) \right) \overline{\pi(g)}$$

$$= \frac{1}{|A|} \sum_{a \in A} \chi_{tr}(a)\overline{\pi(a)} = \dim \pi.$$

□

Theorem 3.16 tells us that for any central extension $1 \to A \to \widetilde{G} \to G \to 1$ and any subgroup $H \subset G$, we have that two elements of $\operatorname{Irr} \widetilde{H}$ which restrict to different characters of A will induce to orthogonal representations on $\widetilde{G}$. An elegant classification result would be a partial converse of this statement. To this end, we present the following conjecture, which is stronger and more natural than Conjecture 1:

Conjecture 2. If $\pi_1, \pi_2 \in \operatorname{Irr}(\widetilde{U})$ are honest representations and for $i = 1, 2$ and some $\chi \in \operatorname{Irr}(A)$,

$$\left\langle \pi_i, \operatorname{Ind}_A^{\widetilde{U}} \chi \right\rangle > 0,$$

then $\operatorname{Ind}_{\widetilde{U}}^{\widetilde{G}} \pi_1 = \operatorname{Ind}_{\widetilde{U}}^{\widetilde{G}} \pi_2$.

This statement seems to be experimentally true in our examples. Notably, it needs the restrictions placed on it: even for the four small G with data listed in Appendix B, this conjecture fails when replacing $\widetilde{U}$ by $\widetilde{B}$. It also fails for inflated representations. This suggests that, if true, this conjecture will require properties of both reductive groups and of $\widetilde{U}$ in particular, making it a tempting and satisfying target for a metaplectic Gelfand–Graev analogue.

Specifically, Conjecture 2 would tell us that representations in $\operatorname{Irr} \widetilde{U}$ could be partitioned into equivalence classes based on restriction to A. Then by selecting exactly one representation π_{χ_i} from each equivalence class, we would have a representation $\oplus_{\chi_i} \pi_{\chi_i}$ on $\widetilde{U}$ which would induce to a representation supported on every element of $\operatorname{Irr} \widetilde{G}$ with minimal multiplicity.

We now turn our attention to better understanding the number and degree of honest representations of $\widetilde{U}$. This information could help prove or modify Conjecture 2. We begin with the following observation:

Remark 4.5. The degree of any representation $\pi \in \operatorname{Irr}(\widetilde{U})$ is a power of p, where p is the characteristic of the field k of the linear algebraic group G.

This fact follows from U and thus $\widetilde{U}$ being p-groups and the fact that the degree of a representation of a group divides the order of the group. The following conjectures, based on the data in Appendix B, attempt to give more information about the degrees of the honest representations of $\widetilde{U}$.

Conjecture 3. Given $\mu, \mu' \in \operatorname{Irr}(\widetilde{U})$, where μ factors through U and μ' doesn't. Then $\dim \mu \le \dim \mu'$.

Conjecture 4. All honest representations of $\widetilde{U}$ have the same degree.

We give a lower bound for the number of honest irreducible representations of $\widetilde{U}$. According to [TT14, Theorem 3.1], for an element $q \in U$, the number of conjugacy classes of $\widetilde{U}$ which map to the conjugacy class of q under the covering map is equal to

$$\#\left\{\chi \text{ irrep. of } A \mid \chi\left(\sigma(q_1, q)\sigma(q, q_1)^{-1}\right) = 1, \ \forall q_1 \in C_U(q)\right\}$$

where $C_U(q)$ is the centralizer of q and $\sigma : G \times G \to A$ is the cocycle. Substituting $q = 1$, we have

$$\#\left\{\chi \text{ irrep. of } A \mid \chi\left(\sigma(q_1, 1)\sigma(1, q_1)^{-1}\right) = 1, \forall q_1 \in Z(1)\right\} = \#\{\chi \text{ irrep. of } A\} = |A|$$

as each Schur multiplier A is the kernel of a central extension and thus abelian. The relation

$$\chi\left(\sigma(q_1, q)\sigma(q, q_1)^{-1}\right) = 1$$

is always satisfied by the trivial representation, so we know that each $\widetilde{U}$ must have at least $|A| - 1$ more conjugacy classes than U, and therefore $\widetilde{U}$ has at least $|A| - 1$ honest representations.

Upon inspection, we find that when G is $A_1(4)$, $A_3(2)$, or $A_1(9)$, $\widetilde{U}$ has exactly $|A| - 1$ honest representations, whereas $A_2(2)$ has one more than this lower bound.

We first make a remark:

Remark 4.6. From its definition, a cocycle σ satisfies the *cocycle condition*

$$a\sigma(b, c) - \sigma(ab, c) + \sigma(a, bc) - \sigma(a, b) = 0$$

for all a, b, c in G, where $a\sigma(b, c)$ is the element $a \in G$ acting on $\sigma(b, c) \in A$ under some group action. As mentioned in [Kar87, page 5], in the case of the Schur multiplier, this action of G on $A = M(G)$ is trivial.

We now observe the following fact:

Proposition 4.7. For all positive integers r and elements $q \in U$, $\sigma(q^r, q) = \sigma(q, q^r)$.

Proof. The cocycle must satisfy $a\sigma(b,c) - \sigma(ab,c) + \sigma(a,bc) - \sigma(a,b) = 0$ for all $a, b, c \in U$. When we set $a = b = c = q$, we have that $q\sigma(q,q) - \sigma(q^2,q) + \sigma(q,q^2) - \sigma(q,q) = 0$. Since U can only act trivially on A as stated in Remark 4.6, we have that $\sigma(q^2,q) = \sigma(q,q^2)$. Proceeding by induction, suppose $\sigma(q^r,q) = \sigma(q,q^r)$. Then $q\sigma(q^r,q) - \sigma(q^{r+1},q) + \sigma(q,q^{r+1}) - \sigma(q,q^r) = 0$. Again, q must act trivially on A, and so our inductive hypothesis gives up $\sigma(q^{r+1},q) = \sigma(q,q^{r+1}$, proving the result. ☐

Corollary 4.7.1. If $C(q) = \langle q \rangle$, then the number of conjugacy classes of $\widetilde{U}$ which map into the conjugacy class of q is $|A|$.

Proof. Using Proposition 4.7, the condition in [TT14, Theorem 3.1] reduces to

$$\#\left\{\chi \text{ irrep. of } A \mid \chi(1) = 1,\ \forall q_1 \in C_U(q)\right\}.$$

The required condition is true for all $\chi \in \mathrm{Irr}(A)$, and thus the size of this set is $|A|$ since A is abelian. ☐

The following conjecture arises from studying the conjugacy classes of the unipotent subgroup U of $A_2(2)$ and hypothesizing that the converse of the above is true.

Conjecture 5. The number of conjugacy classes of $\widetilde{U}$ which map to the conjugacy class of $q \in U$ under the covering map is

$$\begin{cases} |A|, & \text{if } C_U(q) = \langle q \rangle \\ 1, & \text{otherwise.} \end{cases}$$

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Appendix A. GAPpendix

In computing data on induced representations of $\widetilde{U}$, we wrote several useful functions in GAP 4.10 which expedited the process. They are recorded here, along with brief explanations of their functionality.

`IP` specifies the standard inner product on two characters of the same group G. It takes as its arguments `T` the character table of G; `chi` any character of G; and `psi` any positive integer in the range $[1, \# \operatorname{Irr}(G)]$, correponding to the `psi`th irreducible character (under the order in which GAP stores them). `IP` outputs $\langle \chi, \operatorname{Irr}(G)[\psi] \rangle$.

```
IP := function(T, chi, psi)
    local S;
    S := SizesConjugacyClasses(T);
    return Sum( [1 .. Length(S)],
            i -> S[i] * chi[i] * ComplexConjugate(Irr(T)[psi][i]) ) / Size(T);
end;;
```

`coeffs` returns the coefficients $(c_1, \dots, c_n)$ arising as $\chi = \sum c_i \chi_i$ for χ_i the n irreducible characters in character table T (numbered according to GAP's internal system).

```
coeffs := function(T, chi)
    return List( [1..Length(SizesConjugacyClasses(T))], i -> IP(T, chi, i) );
end;;
```

`embed` returns a group $U' \leqslant H$ such that $U' \cong U$, given $U \leqslant G \cong H$.

```
embed := function(U, G, H)
    local phi, gens, image_gens, isom, proj;
    phi := IsomorphismGroups(G, H);
    gens := GeneratorsOfGroup(G);
    image_gens := List( gens, x -> Image(phi, x) );
    isom := GroupHomomorphismByImages(G, H, gens, image_gens);
    return Image(isom, U);
end;;
```

Given groups G and H admitting the projection map $H \xrightarrow{p} G$, as well as $U \leqslant G$, `lift` returns $p^{-1}(U) \leqslant H$.

```
lift := function(U, G, H)
    local proj;
    proj := GroupHomomorphismByImages(H, G);
    return PreImage(proj, U);
end;;
```

`IC` returns the induced characters $\operatorname{Ind}_G^H \chi$ for each χ an irreducible character of G.

```
IC := function(G, H)
    local n, ic, ctg;
    ic := [ ];
    ctg := CharacterTable(G);
    for n in [1..Length(Irr(G))] do
        Add(ic, InducedClassFunction(ClassFunction(ctg, Irr(G)[n]), H));
    od;
    return ic;
end;;
```

`charsAppearing` returns, for each irreducible character χ of G, the indices of the irreducible characters θ of H for which $\left\langle \operatorname{Ind}_G^H \chi, \theta \right\rangle_H \neq 0$. (`charsAppearing` is purely a function for intuitive convenience and it carries strictly less information than `charCoeffs`.)

```
charsAppearing := function(G, H)
    local ic, i, m, n;
    ic := IC(G, H);
    for i in [1..Length(Irr(G))] do
        if i < 10 then Print(" "); fi; Print(i); Print(": ");
        for m in [1..Length(ConstituentsOfCharacter(ic[i]))] do
            for n in [1..Length(Irr(H))] do
                if ConstituentsOfCharacter(ic[i])[m] = Irr(H)[n] then
                    Print(n); Print(" "); break;
```

```
                fi;
            od;
        od; Print("\n");
    od;
end;;
```

`charsAppearing` returns, for each irreducible character χ of G, the values $\left\langle \mathrm{Ind}_G^H \chi, \theta \right\rangle_H \neq 0$ for each irreducible character θ of H.

```
charCoeffs := function(G, H)
    local ic, i, x, cth;
    ic := IC(G, H);
    cth := CharacterTable(H);
    for i in [1..Length(ic)] do
        if i < 10 then Print(" "); fi; Print(i); Print(": ");
        for x in [1..Length(coeffs(cth, ic[i]))] do
            if coeffs(cth, ic[i])[x] = 0
                then Print(". ");
                else Print(coeffs(cth, ic[i])[x]); Print(" ");
            fi;
        od; Print("\n");
    od;
end;;
```

Given the nth irreducible character χ_n of G (according to GAP's internal numbering), for each 1-dimensional irreducible character χ of G, `1twistSum` returns the values $\left\langle \mathrm{Ind}_G^H(\chi_n \otimes \chi), \theta \right\rangle$ for each irreducible character θ of H.

```
1twistSum := function(G, H, n)
    local ic, rep, 1dims, twists, cth, i, m, x;
    ic := IC(G, H);
    rep := ic[n];
    1dims := [ ];
    for x in Irr(H) do
        if x[1] = 1 then Add(1dims, x); fi;
    od;
    twists := List(1dims, x -> rep * x);
    cth := CharacterTable(H);
    for i in [1..Length(twists)] do
        if i < 10 then Print(" "); fi; Print(i); Print(": ");
        for m in [1..Length(coeffs(cth, twists[i]))] do
            if coeffs(cth, twists[i])[m] = 0
                then Print(". ");
                else Print(coeffs(cth, twists[i])[m]); Print(" ");
            fi;
        od; Print("\n");
    od;
end;;
```

For each 1-dimensional character χ of G, for each irreducible character θ of G with $\dim\theta > 1$, `1twistIrreps` index n (according to GAP's internal numbering) of $\chi_n = \chi \otimes \theta$.

```
1twistIrreps := function(G)
    local 1dims, x, i, n, y;
    1dims := [ ];
    for x in Irr(G) do
        if x[1] = 1 then Add(1dims, x); fi;
    od;
    for x in [1..Length(1dims)] do
        if x < 10 then Print(" "); fi;
        Print(x); Print(": ");
        for i in [Length(1dims)+1 .. Length(Irr(G))] do
            y := 1dims[x] * Irr(G)[i];
            for n in [Length(1dims)+1 .. Length(Irr(G))] do
                if Irr(G)[n] = y then
                    if n < 10 then Print(" "); fi;
```

	$\lvert U \rvert$ 1	$\lvert U \rvert$ 2	$\lvert U \rvert$ 2	$\lvert U \rvert$ 2
χ_1	1	1	1	1
χ_2	1	−1	−1	1
χ_3	1	−1	1	−1
χ_4	1	1	−1	−1

FIGURE 1. For $G = SL_2(\mathbb{F}_4)$, `Display(CharacterTable(U));`.

	$\lvert\widetilde{U}\rvert$ 1	4 4	4 4	4 4	$\lvert\widetilde{U}\rvert$ 2
χ_1	1	1	1	1	1
χ_2	1	−1	1	−1	1
χ_3	1	1	−1	−1	1
χ_4	1	−1	−1	1	1
χ_5	2				−2

FIGURE 2. For $G = SL_2(\mathbb{F}_4)$, `Display(CharacterTable(Utilde));`.

```
                    Print(n); Print(" "); break;
                fi;
            od;
        od; Print("\n");
    od;
end;;
```

APPENDIX B. CHARACTER TABLES AND OTHER DATA

In each character table, the first row in each column shows the size of the centralizer of that conjugacy class; in particular, it is noted when this size is the size of the group. The second row shows the order of each element in that class.

In the context of GAP code, $[n, m]$ denotes the m-th group of order n, i.e. `SmallGroup(n,m)`.

Case 1: $G = A_1(4)$. We have the isomorphisms $A_1(4) \cong SL_2(\mathbb{F}_4) \cong PSL_2(\mathbb{F}_4) \cong PSL_2(\mathbb{F}_5) \cong A_5$, the former via definition; the next via $\operatorname{char}\mathbb{F}_4 = 2$; and the latter two via [ATLAS]. We have the additional isomorphisms $\widetilde{G} \cong SL_2(\mathbb{F}_5)$ [ATLAS], $U \cong V_4$, $\widetilde{U} \cong Q_8$ the quaternion group, $B \cong V_4 \rtimes C_3$, and $\widetilde{B} \cong Q_8 \rtimes C_3$.

Define the following in GAP:

```
> m1 := [ [ Z(4)^0 , Z(4) ] , [ Z(4)*0, Z(4)^0] ];;
> m2 := [ [ Z(4)^0 , Z(4) + Z(4)^0 ] , [ Z(4)*0, Z(4)^0] ];;
> m3 := [ [ Z(4) , Z(4)*0 ] , [ Z(4)*0, Z(4)^2 ] ];
> G := SL(2,4);;
> A5 := AlternatingGroup(5);;
> Gtilde := SchurCover(A5);;
> U := embed(Group(m1,m2), G, A5);;
> Utilde := lift(U, A5, Gtilde);;
> B := embed(Group(m1,m2,m3), G, A5);;
> Btilde := lift(B, A5, Gtilde);;
```

In GAP, G is stored as $[60, 5]$, $\widetilde{G}$ as $[120, 5]$, U as $[4, 2]$, $\widetilde{U}$ as $[8, 4]$, B as $[12, 3]$, and $\widetilde{B}$ as $[24, 3]$.

See Figures 1–6 for the relevant character tables.

Case 2: $G = A_2(2)$. We have the isomorphisms $A_2(2) \cong SL_3(\mathbb{F}_2) \cong PSL_3(\mathbb{F}_2) \cong PSL_2(\mathbb{F}_7)$, the former via definition; the intermediate via $\operatorname{char}\mathbb{F}_2 = 2$; and the latter via [ATLAS]. We have the additional isomorphisms $U = B \cong D_8$ and $\widetilde{U} = \widetilde{B} \cong Q_{16}$ (with the former equalities because $\operatorname{char}\mathbb{F}_2 = 2$).

Define the following in GAP:

```
> m1 := [ [ Z(2), Z(2), 0*Z(2) ], [ 0*Z(2), Z(2), 0*Z(2) ], [ 0*Z(2), 0*Z(2), Z(2) ] ];;
> m2 := [ [ Z(2), 0*Z(2), Z(2) ], [ 0*Z(2), Z(2), 0*Z(2) ], [ 0*Z(2), 0*Z(2), Z(2) ] ];;
> m3 := [ [ Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2), Z(2) ], [ 0*Z(2), 0*Z(2), Z(2) ] ];;
> G := SL(3,2);;
```

	$\lvert B\rvert$	4	3	3
	1	2	3	3
χ_1	1	1	1	1
χ_2	1	1	a	$\frac{1}{a}$
χ_3	1	1	$\frac{1}{a}$	a
χ_4	3	-1		

FIGURE 3. For $G = SL_2(\mathbb{F}_4)$, `Display(CharacterTable(B));`, for $a = \frac{-1+i\sqrt{3}}{2}$.

	$\lvert\widetilde{B}\rvert$	6	6	6	6	4	$\lvert\widetilde{B}\rvert$
	$1a$	$6a$	$3a$	$6b$	$3b$	$4a$	$2a$
χ_1	1	1	1	1	1	1	1
χ_2	1	a	$\frac{1}{a}$	$\frac{1}{a}$	a	1	1
χ_3	1	$\frac{1}{a}$	a	a	$\frac{1}{a}$	1	1
χ_4	2	1	-1	1	-1		-2
χ_5	2	a	$-\frac{1}{a}$	$\frac{1}{a}$	$-a$		-2
χ_6	2	$\frac{1}{a}$	$-a$	a	$-\frac{1}{a}$		-2
χ_7	3					-1	3

FIGURE 4. For $G = SL_2(\mathbb{F}_4)$, `Display(CharacterTable(Btilde));`, for $a = \frac{-1+i\sqrt{3}}{2}$.

	$\lvert G\rvert$	4	5	5	3
	1	2	5	5	3
χ_1	1	1	1	1	1
χ_2	3	-1	a	$\overline{a}$	
χ_3	3	-1	$\overline{a}$	a	
χ_4	4		-1	-1	1
χ_5	5	1			-1

FIGURE 5. For $G = SL_2(\mathbb{F}_4)$, `Display(CharacterTable(G));`, for $a = \frac{1-i\sqrt{5}}{2}$.

	$\lvert\widetilde{G}\rvert$	10	10	6	10	10	4	6	$\lvert\widetilde{G}\rvert$
	1	10	10	6	5	5	4	3	2
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	a	$\overline{a}$	1	$-\overline{a}$	$-a$		-1	-2
χ_3	2	$\overline{a}$	a	1	$-a$	$-\overline{a}$		-1	-2
χ_4	3	$\overline{a}$	a		a	$\overline{a}$	-1		3
χ_5	3	a	$\overline{a}$		$\overline{a}$	a	-1		3
χ_6	4	-1	-1	1	-1	-1		1	4
χ_7	4	1	1	-1	-1	-1		1	-4
χ_8	5			-1			1	-1	5
χ_9	6	-1	-1		1	1			-6

FIGURE 6. For $G = SL_2(\mathbb{F}_4)$, `Display(CharacterTable(Gtilde));`, for $a = \frac{1-i\sqrt{5}}{2}$.

	χ_1	χ_2	χ_3	χ_4
θ_1	1	1	1	
θ_2				1
θ_3				1
θ_4				1

FIGURE 7. For $G = SL_2(\mathbb{F}_9)$, `charCoeffs(U, B);`.

	χ_1	χ_2	χ_3	χ_4	χ_5
θ_1	1			1	
θ_2					1
θ_3					1
θ_4		1	1	1	1

FIGURE 8. For $G = SL_2(\mathbb{F}_9)$, `charCoeffs(B, G);`.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
θ_1	1	1	1				
θ_2							1
θ_3							1
θ_4							1
θ_5				1	1	1	

FIGURE 9. For $G = SL_2(\mathbb{F}_9)$, `charCoeffs(Utilde, Btilde);`.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
θ_1	1					1			
θ_2								1	
θ_3								1	
θ_4		1	1						1
θ_5							1		1
θ_6							1		1
θ_7				1	1	1		1	

FIGURE 10. For $G = SL_2(\mathbb{F}_9)$, `charCoeffs(Btilde, Gtilde);`.

	$\lvert U\rvert$ 1	4 2	$\lvert U\rvert$ 2	4 2	4 4
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	1	-1	-1
χ_4	1	-1	1	1	-1
χ_5	2		-2		

FIGURE 11. For $G = SL_3(\mathbb{F}_2)$, `Display(CharacterTable(U));`.

```
> Gtilde := SchurCover(G);;
> U := Group(m1,m2,m3);;
> Utilde := lift(U, G, Gtilde);;
```

In GAP, `G` is stored as $[168, 42]$, `Gtilde` as $[336, 114]$, `U` as $[8, 3]$, and `Utilde` as $[16, 9]$. See Figures 11–14 for the relevant character tables.

Case 3: $G = A_3(2)$. We have the isomorphisms $A_3(2) \cong SL_4(\mathbb{F}_2) \cong PSL_4(\mathbb{F}_2) \cong A_8$, the former via definition; the intermediate via $\operatorname{char} \mathbb{F}_2 = 2$; and the latter via [ATLAS]. The 2-fold Schur cover of G can be seen as the preimage of $A_8 \leqslant SO(7)$—permuting components of the subspace of $\mathbb{R}^8$ with components summing to 0—under the 2-fold cover $\mathrm{Spin}(7) \longrightarrow SO(7)$.

Define the following in GAP:

```
> m1 := [ [ Z(2)^0, Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ], [ 0*Z(2)
    , 0*Z(2), Z(2)^0, 0*Z(2) ], [0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ] ];;
> m2 := [ [ Z(2)^0, 0*Z(2), Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ], [ 0*Z(2)
    , 0*Z(2), Z(2)^0, 0*Z(2) ], [0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ] ];;
> m3 := [ [ Z(2)^0, 0*Z(2), 0*Z(2), Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ], [ 0*Z(2)
    , 0*Z(2), Z(2)^0, 0*Z(2) ], [0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ] ];;
> m4 := [ [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, Z(2), 0*Z(2) ], [ 0*Z(2)
    , 0*Z(2), Z(2)^0, 0*Z(2) ], [0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ] ];;
```

	$\lvert\widetilde{U}\rvert$ 1	4 4	$\lvert\widetilde{U}\rvert$ 2	4 4	8 8	8 8	8 4
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	1
χ_3	1	-1	1	1	-1	-1	1
χ_4	1	1	1	-1	-1	-1	1
χ_5	2		2				-2
χ_6	2		-2		$-\sqrt{2}$	$\sqrt{2}$	
χ_7	2		-2		$\sqrt{2}$	$-\sqrt{2}$	

FIGURE 12. For $G = SL_3(\mathbb{F}_2)$, `Display(CharacterTable(Utilde));`.

	$\lvert G\rvert$ 1	8 2	4 4	7 7	7 7	3 3
χ_1	1	1	1	1	1	1
χ_2	3	-1	1	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{4}$	
χ_3	3	-1	1	$\frac{-1-i\sqrt{7}}{4}$	$\frac{-1+i\sqrt{7}}{2}$	
χ_4	6	2		-1	-1	
χ_5	7	-1	-1			1
χ_6	8			1	1	-1

FIGURE 13. For $G = SL_3(\mathbb{F}_2)$, `Display(CharacterTable(G));`.

	$\lvert\widetilde{G}\rvert$ 1	6 3	14 7	14 7	8 4	6 6	8 8	14 14	8 8	14 14	$\lvert\widetilde{G}\rvert$ 2
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3		a	$\frac{1}{a}$	-1		1	a	1	$\frac{1}{a}$	3
χ_3	3		$\frac{1}{a}$	a	-1		1	$\frac{1}{a}$	1	a	3
χ_4	4	1	$-a$	$\frac{1}{a}$		-1		a		$\frac{1}{a}$	-4
χ_5	4	1	$\frac{1}{a}$	$-a$		-1		$\frac{1}{a}$		a	-4
χ_6	6		-1	-1	2			-1		-1	6
χ_7	6		-1	-1			$-\sqrt{2}$	1	$\sqrt{2}$	1	-6
χ_8	6		-1	-1			$\sqrt{2}$	1	$-\sqrt{2}$	1	-6
χ_9	7	1			-1	1	-1		-1		7
χ_{10}	8	-1	1	1		-1		1		1	8
χ_{11}	8	-1	1	1		1		-1		-1	-8

FIGURE 14. For $G = SL_3(\mathbb{F}_2)$, `Display(CharacterTable(Gtilde));`, for $a = \frac{-1+i\sqrt{7}}{2}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
θ_1	1			2		1
θ_2		1	1		1	1
θ_3				1	1	1
θ_4				1	1	1
θ_5		1	1	1	2	2

FIGURE 15. For $G = SL_3(\mathbb{F}_2)$, `charCoeffs(U, G);`, where each row shows the values c_j for $\operatorname{Ind}_U^G \theta_i = \sum_j c_j \chi_j$.

```
> m5 := [ [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2), Z(2) ], [ 0*Z(2)
    , 0*Z(2), Z(2)^0, 0*Z(2) ], [0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ] ];;
> m6 := [ [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ], [ 0*Z
    (2), 0*Z(2), Z(2)^0, Z(2) ], [0*Z(2), 0*Z(2), 0*Z(2), Z(2)^0 ] ];;
> G := SL(4,2);;
> Gtilde := SchurCover(G);;
```

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
θ_1	1					2				1	
θ_2		1	1						1	1	
θ_3						1			1	1	
θ_4						1			1	1	
θ_5		1	1			1			2	2	
θ_6				1	1		1	2			2
θ_7				1	1		2	1			2

FIGURE 16. For $G = SL_3(\mathbb{F}_2)$, `charCoeffs(Utilde, Gtilde);`, where each row shows the values c_j for $\mathrm{Ind}_{\widetilde{U}}^{\widetilde{G}}\theta_i = \sum_j c_j\chi_j$.

	$\lvert U\rvert$	16	32	16	8	$\lvert U\rvert$	16	32	16	32	16	16	16	16	8	8
	1	2	2	2	4	2	2	2	4	2	2	2	4	4	4	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	−1	1	1	−1	1	−1	1	−1	1	−1	1	1	−1
χ_3	1	1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	−1	1	1
χ_4	1	−1	1	1	−1	1	1	1	−1	1	−1	1	−1	1	−1	1
χ_5	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_6	1	−1	1	−1	1	1	−1	1	−1	1	1	−1	1	−1	−1	1
χ_7	1	1	1	−1	−1	1	−1	1	1	1	1	1	1	1	−1	−1
χ_8	1	−1	1	1	−1	1	1	1	−1	1	1	−1	1	−1	1	−1
χ_9	2		−2			2		2		−2	−2		2			
χ_{10}	2		−2			2		2		−2	2		−2			
χ_{11}	2		−2			2		−2		2		−2		2		
χ_{12}	2		−2			2		−2		2		2		−2		
χ_{13}	2	−2	2			2		−2	2	−2						
χ_{14}	2	2	2			2		−2	−2	−2						
χ_{15}	4			−2		−4	2									
χ_{16}	4			2		−4	−2									

FIGURE 17. For $G = SL_4(\mathbb{F}_2)$, `Display(CharacterTable(U));`.

```
> U := Group(m1,m2,m3,m4,m5,m6);;
> Utilde := lift(U, G, Gtilde);;
```

In GAP, `U` is stored as $[64, 138]$ and `Utilde` as $[128, 931]$.
See Figures 17–20 for the relevant character tables.

Case 4: $G = A_1(9)$. We have the isomorphism $A_1(9) \cong SL_2(\mathbb{F}_9)$ by definition. From [ATLAS], G's 3-fold Schur cover $\widetilde{G}$ is isomorphic to $6.A6$, the 6-fold Schur cover of A_6. Further, $U \cong C_3 \times C_3$, $\widetilde{U} \cong (C_3 \times C_3) \rtimes C_3$, $B \cong (C_3 \times C_3) \rtimes C_8$, and $\widetilde{B} \cong ((C_3 \times C_3) \rtimes C_3) \rtimes C_8$.
Define the following in GAP:

```
> m1 := [ [ Z(9)^0, Z(9)^0 ], [ Z(9)*0, Z(9)^0 ] ];;
> m2 := [ [ Z(9)^0, Z(9)^2 ], [ Z(9)*0, Z(9)^0 ] ];;
> m3 := [ [ Z(9)^0, Z(9)*0 ], [ Z(9)*0, Z(9)^7 ] ];;
> G := SL(2,9);;
> Gtilde := SchurCover(G);;
> U := Group(m1,m2);;
> Utilde := list(U, G, Gtilde);;
> B := Group(m1,m2,m3);;
> Btilde := lift(B, G, Gtilde);;
```

In GAP, G is stored as $[720, 409]$, U as $[9, 2]$, $\widetilde{U}$ as $[27, 3]$, B as $[72, 19]$, and $\widetilde{B}$ as $[216, 25]$.
See Figures 23–28 for the relevant character tables.

Other exceptional groups. An attempt to check Conjecture 5 against other groups resulted in an error from GAP, which remains in effect for the package Unipot 1.4 and GAP 4.10. This error may be patched using [Hor], giving the following commands:

	$\|\widetilde{U}\|$	16	8	8	16	16	32	16	64	8	32	32	$\|\widetilde{U}\|$	16	16	16	16
	1	4	4	4	4	2	4	4	2	8	2	2	2	2	4	2	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	1	1	−1	1	1	1	−1	1	1	1	−1	−1	−1	−1
χ_3	1	1	−1	−1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1
χ_4	1	−1	−1	1	−1	−1	1	−1	1	1	1	1	1	−1	1	1	−1
χ_5	1	1	1	−1	−1	1	1	−1	1	−1	1	1	1	−1	1	1	−1
χ_6	1	−1	1	−1	−1	−1	1	−1	1	1	1	1	1	1	−1	−1	1
χ_7	1	1	−1	1	−1	1	1	−1	1	−1	1	1	1	1	−1	−1	1
χ_8	1	−1	−1	−1	1	−1	1	1	1	−1	1	1	1	1	1	1	1
χ_9	2						−2		2		2	−2	2		2	−2	
χ_{10}	2						−2		2		2	−2	2		−2	2	
χ_{11}	2						−2		2		−2	2	2	−2			2
χ_{12}	2				−2		2	2	2		−2	−2	2				
χ_{13}	2				2		2	−2	2		−2	−2	2				
χ_{14}	2						−2		2		−2	2	2	2			−2
χ_{15}	4	2				−2			−4				4				
χ_{16}	4	−2				2			−4				4				
χ_{17}	8												−8				

FIGURE 18. For $G = SL_4(\mathbb{F}_2)$, `Display(CharacterTable(Utilde));`.

	$\|G\|$	192	96	16	8	180	12	15	15	15	18	6	7	7
	1	2	2	4	4	3	6	15	15	5	3	6	7	7
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	−1	3	−1	1	4		−1	−1	2	1	−1		
χ_3	14	6	2	2		−1	−1	−1	−1	−1	2			
χ_4	20	4	4			5	1				−1	1	−1	−1
χ_5	21	−3	1	1	−1	6	−2	1	1	1				
χ_6	21	−3	1	1	−1	−3	1	a	$\frac{1}{a}$	1				
χ_7	21	−3	1	1	−1	−3	1	$\frac{1}{a}$	a	1				
χ_8	28	−4	4			1	1	1	1	−2	1	−1		
χ_9	35	3	−5	−1	−1	5	1				2			
χ_{10}	45	−3	−3	1	1								b	$\frac{1}{b}$
χ_{11}	45	−3	−3	1	1								$\frac{1}{b}$	b
χ_{12}	56	8				−4		1	1	1	−1	−1		
χ_{13}	64					4		−1	−1	−1	−2		1	1
χ_{14}	70	−2	2	−2		−5	−1				1	1		

FIGURE 19. For $G = SL_4(\mathbb{F}_2)$, `Display(CharacterTable(G));`, for $a = \frac{-1+i\sqrt{15}}{2}$ and $b = -\frac{1+i\sqrt{7}}{2}$.

```
diff --git a/lib/unipot.gi b/lib/unipot.gi
index 103b67c..0339655 100644
--- a/lib/unipot.gi
+++ b/lib/unipot.gi
@@ -435,7 +435,7 @@ InstallOtherMethod( GeneratorsOfGroup,
            ListX( [1 .. Length(PositiveRoots(RootSystem(U)))],
                    Difference( AsSet( Fam!.ring ), [Zero( Fam!.ring )] ),
                    function( r, x )
-                          return UnipotChevElem( U, rec(roots:=r, felems:=x),
    UNIPOT_DEFAULT_REP );
+                          return UnipotChevElem( U, rec(roots:=[r], felems:=[x]),
    UNIPOT_DEFAULT_REP );
                    end
                  );
      end
```

	$\lvert\widetilde{G}\rvert$ 1	$\lvert\widetilde{G}\rvert$ 2	30 5	30 10	360 6	360 3	30 15	30 30	30 15	30 30	192 2	16 4	96 4	8 8	12 12	36 6	36 3	12 6	12 6	14 7	14 14	14 7	14 14
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	7	2	2	4	4	-1	-1	-1	-1	-1	-1	3	1		1	1	-1	-1				
χ_3	8	-8	-2	2	4	-4	1	-1	1	-1						-2	2			1	-1	1	-1
χ_4	14	14	-1	-1	-1	-1	-1	-1	-1	-1	6	2	2		-1	2	2						
χ_5	20	20			5	5					4		4		1	-1	-1	1	1	-1	-1	-1	-1
χ_6	21	21	1	1	6	6	1	1	1	1	-3	1	1	-1	-2								
χ_7	21	21	1	1	-3	-3	a	a	$\frac{1}{a}$	$\frac{1}{a}$	-3	1	1	-1	1								
χ_8	21	21	1	1	-3	-3	$\frac{1}{a}$	$\frac{1}{a}$	a	a	-3	1	1	-1	1								
χ_9	24	-24	-1	1	6	-6	-1	1	-1	1										c	$-c$	c	$-\frac{1}{c}$
χ_{10}	24	-24	-1	1	6	-6	-1	1	-1	1										$\frac{1}{c}$	$-\frac{1}{c}$	c	$-c$
χ_{11}	28	28	-2	-2	1	1	1	1	1	1	-4		4		1	1	1	-1	-1				
χ_{12}	35	35			5	5					3	-1	-5	-1	1	2	2						
χ_{13}	45	45									-3	1	-3	1						c	c	$\frac{1}{c}$	$\frac{1}{c}$
χ_{14}	45	45									-3	1	-3	1						$\frac{1}{c}$	$\frac{1}{c}$	c	c
χ_{15}	48	-48	-2	2	-6	6	1	-1	1	-1										-1	1	-1	1
χ_{16}	56	56	1	1	-4	-4	1	1	1	1	8					-1	-1	-1	-1				
χ_{17}	56	-56	1	-1	-2	2	a	$-a$	$\frac{1}{a}$	$-\frac{1}{a}$						-2	2						
χ_{18}	56	-56	1	-1	-2	2	$\frac{1}{a}$	$-\frac{1}{a}$	a	$-a$						-2	2						
χ_{19}	56	-56	1	-1	4	-4	1	-1	1	-1						1	-1	b	$-b$				
χ_{20}	56	-56	1	-1	4	-4	1	-1	1	-1						1	-1	$-b$	b				
χ_{21}	64	64	-1	-1	4	4	-1	-1	-1	-1						-2	-2			1	1	1	1
χ_{22}	64	-64	-1	1	-4	4	-1	1	-1	1						2	-2			1	-1	1	-1
χ_{23}	70	70			-5	-5					-2	-2	2		-1	1	1	1	1				

FIGURE 20. For $G = SL_4(\mathbb{F}_2)$, `Display(CharacterTable(Gtilde));`, for $a = -\frac{1+i\sqrt{15}}{2}$, $b = -i\sqrt{3}$, and $c = -\frac{1+i\sqrt{7}}{2}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}
θ_1	1		3	2								3	1	
θ_2					1	1	1	1		1	1		1	1
θ_3									1	1	1	1	1	1
θ_4		1		1				1				1	1	2
θ_5			1	1					1			2	1	1
θ_6									1	1	1	1	1	1
θ_7			1	1					1			2	1	1
θ_8			1	1					1			2	1	1
θ_9					1	1	1	1	1	2	2	1	2	2
θ_{10}			1	1					2	1	1	3	2	2
θ_{11}					1	1	1	1	1	2	2	1	2	2
θ_{12}		1		1	1	1	1	2		1	1	1	2	3
θ_{13}					1	1	1	1	1	2	2	1	2	2
θ_{14}			1	1					2	1	1	3	2	2
θ_{15}		1		1	2	2	2	3	1	3	3	2	4	5
θ_{16}			1	1	1	1	1	1	3	3	3	4	4	4

FIGURE 21. For $G = SL_4(\mathbb{F}_2)$, `charCoeffs(U, G);`, where each row shows the values c_j for $\mathrm{Ind}_U^G \theta_i = \sum_j c_j \chi_j$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}
θ_1	1			3	2											3					1		
θ_2						1	1	1			1		1	1							1		1
θ_3		1			1						1					1					1		2
θ_4												1	1	1		1					1		1
θ_5				1	1							1				2					1		1
θ_6												1	1	1		1					1		1
θ_7				1	1							1				2					1		1
θ_8				1	1							1				2					1		1
θ_9						1	1	1			1	1	2	2		1					2		2
θ_{10}				1	1							2	1	1		3					2		2
θ_{11}						1	1	1			1	1	2	2		1					2		2
θ_{12}		1			1	1	1	1			2		1	1		1					2		3
θ_{13}						1	1	1			1	1	2	2		1					2		2
θ_{14}				1	1							2	1	1		3					2		2
θ_{15}		1			1	2	2	2			3	1	3	3		2					4		5
θ_{16}				1	1	1	1	1			1	3	3	3		4					4		4
θ_{17}			1						3	3					6		7	7	7	7		8	

FIGURE 22. For $G = SL_4(\mathbb{F}_2)$, `charCoeffs(Utilde, Gtilde);`, where each row shows the values c_j for $\operatorname{Ind}_{\widetilde{U}}^{\widetilde{G}} \theta_i = \sum_j c_j \chi_j$.

	$\lvert U\rvert$ 1	$\lvert U\rvert$ 3	$\lvert U\rvert$ 3	$\lvert U\rvert$ 3	$\lvert U\rvert$ 3	$\lvert U\rvert$ 3	$\lvert U\rvert$ 3	$\lvert U\rvert$ 3	$\lvert U\rvert$ 3
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	a	a	$\frac{1}{a}$	$\frac{1}{a}$	$\frac{1}{a}$	a
χ_3	1	1	1	$\frac{1}{a}$	$\frac{1}{a}$	a	a	a	$\frac{1}{a}$
χ_4	1	a	$\frac{1}{a}$	1	a	a	1	$\frac{1}{a}$	$\frac{1}{a}$
χ_5	1	$\frac{1}{a}$	a	1	$\frac{1}{a}$	$\frac{1}{a}$	1	a	a
χ_6	1	a	$\frac{1}{a}$	a	$\frac{1}{a}$	1	$\frac{1}{a}$	a	1
χ_7	1	$\frac{1}{a}$	a	$\frac{1}{a}$	$\frac{1}{a}$	1	a	$\frac{1}{a}$	1
χ_8	1	a	$\frac{1}{a}$	$\frac{1}{a}$	1	$\frac{1}{a}$	a	1	a
χ_9	1	$\frac{1}{a}$	a	a	1	a	$\frac{1}{a}$	1	$\frac{1}{a}$

FIGURE 23. For $G = SL_2(\mathbb{F}_9)$, `Display(CharacterTable(U));`, for $a = -\frac{1+i\sqrt{3}}{2}$.

	$\lvert\widetilde{U}\rvert$ 1	9 3	9 3	9 3	9 3	9 3	9 3	9 3	9 3	$\lvert\widetilde{U}\rvert$ 3	$\lvert\widetilde{U}\rvert$ 3
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	a	$\frac{1}{a}$	a	a	1	$\frac{1}{a}$	$\frac{1}{a}$	1	1	1
χ_3	1	$\frac{1}{a}$	a	$\frac{1}{a}$	$\frac{1}{a}$	1	a	a	1	1	1
χ_4	1	$\frac{1}{a}$	a	1	a	$\frac{1}{a}$	$\frac{1}{a}$	1	a	1	1
χ_5	1	a	$\frac{1}{a}$	1	$\frac{1}{a}$	a	a	1	$\frac{1}{a}$	1	1
χ_6	1	1	1	a	$\frac{1}{a}$	$\frac{1}{a}$	a	$\frac{1}{a}$	a	1	1
χ_7	1	1	1	$\frac{1}{a}$	a	a	$\frac{1}{a}$	a	$\frac{1}{a}$	1	1
χ_8	1	a	$\frac{1}{a}$	$\frac{1}{a}$	1	$\frac{1}{a}$	1	a	a	1	1
χ_9	1	$\frac{1}{a}$	a	a	1	a	1	$\frac{1}{a}$	$\frac{1}{a}$	1	1
χ_{10}	3									b	$\frac{1}{b}$
χ_{11}	3									$\frac{1}{b}$	b

FIGURE 24. For $G = SL_2(\mathbb{F}_9)$, `Display(CharacterTable(Utilde));`, for $a = \frac{-1+i\sqrt{3}}{2}$ and $b = -3 \cdot \frac{1+i\sqrt{3}}{2}$.

	$\lvert B\rvert$	18	18	$\lvert B\rvert$	18	18	8	8	8	8	8	8
	1	3	3	2	6	6	8	4	8	8	4	8
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-1	1	-1	-1	1	-1
χ_3	1	1	1	-1	-1	-1	a	$-i$	$-\frac{1}{a}$	$-a$	i	$\frac{1}{a}$
χ_4	1	1	1	-1	-1	-1	$-\frac{1}{a}$	i	a	$\frac{1}{a}$	$-i$	$-a$
χ_5	1	1	1	-1	-1	-1	$\frac{1}{a}$	i	$-a$	$-\frac{1}{a}$	$-i$	a
χ_6	1	1	1	-1	-1	-1	$-a$	$-i$	$\frac{1}{a}$	a	i	$-\frac{1}{a}$
χ_7	1	1	1	1	1	1	i	-1	$-i$	i	-1	$-i$
χ_8	1	1	1	1	1	1	$-i$	-1	i	$-i$	-1	i
χ_9	4	-2	1	-4	2	-1						
χ_{10}	4	-2	1	4	-2	1						
χ_{11}	4	1	-2	-4	-1	2						
χ_{12}	4	1	-2	4	1	-2						

FIGURE 25. For $G = SL_2(\mathbb{F}_9)$, `Display(CharacterTable(B));`, for $a = \frac{-1+i}{\sqrt{2}}$.

	$\lvert\widetilde{B}\rvert$	24	24	24	24	24	24	18	12	18	24	24	24	24	24	24	24	24	18	$\lvert\widetilde{B}\rvert$	24	24	18	24	$\lvert\widetilde{B}\rvert$	$\lvert\widetilde{B}\rvert$	$\lvert\widetilde{B}\rvert$	$\lvert\widetilde{B}\rvert$
	1	8	24	24	24	24	4	4	3	12	6	12	12	12	8	8	24	24	6	2	8	24	3	24	6	3	3	6
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	-1	1	1	1	1
χ_3	1	a	a	a	$-a$	$-a$	$-i$	i	1	i	-1	$-i$	$-i$	i	$-a$	$-\frac{1}{a}$	$-\frac{1}{a}$	$\frac{1}{a}$	-1	-1	$\frac{1}{a}$	$\frac{1}{a}$	1	$-\frac{1}{a}$	-1	1	1	-1
χ_4	1	$-a$	$-a$	$-a$	a	a	$-i$	i	1	i	-1	$-i$	$-i$	i	a	$\frac{1}{a}$	$\frac{1}{a}$	$-\frac{1}{a}$	-1	-1	$-\frac{1}{a}$	$-\frac{1}{a}$	1	$\frac{1}{a}$	-1	1	1	-1
χ_5	1	$-\frac{1}{a}$	$-\frac{1}{a}$	$-\frac{1}{a}$	$\frac{1}{a}$	$\frac{1}{a}$	i	$-i$	1	$-i$	-1	i	i	$-i$	$\frac{1}{a}$	a	a	$-a$	-1	-1	$-a$	$-a$	1	a	-1	1	1	-1
χ_6	1	$\frac{1}{a}$	$\frac{1}{a}$	$\frac{1}{a}$	$-\frac{1}{a}$	$-\frac{1}{a}$	i	$-i$	1	$-i$	-1	i	i	$-i$	$-\frac{1}{a}$	$-a$	$-a$	a	-1	-1	a	a	1	$-a$	-1	1	1	-1
χ_7	1	$-i$	$-i$	$-i$	$-i$	$-i$	-1	-1	1	-1	1	-1	-1	-1	$-i$	i	i	i	1	1	i	i	1	i	1	1	1	1
χ_8	1	i	i	i	i	i	-1	-1	1	-1	1	-1	-1	-1	i	$-i$	$-i$	$-i$	1	1	$-i$	$-i$	1	$-i$	1	1	1	1
χ_9	3	$-\frac{1}{a}$	c	d	$-c$	$-d$	$-i$	i		$-f$		$-\frac{1}{f}$	f	$\frac{1}{f}$	$\frac{1}{a}$	a	$-\frac{1}{c}$	$\frac{1}{d}$		-3	$-a$	$\frac{1}{c}$		$-\frac{1}{d}$	g	$-\frac{1}{g}$	$-g$	$\frac{1}{g}$
χ_{10}	3	$-\frac{1}{a}$	d	c	$-d$	$-c$	$-i$	i		$\frac{1}{f}$		f	$-\frac{1}{f}$	$-f$	$\frac{1}{a}$	a	$-\frac{1}{d}$	$\frac{1}{c}$		-3	$-a$	$\frac{1}{d}$		$-\frac{1}{c}$	$\frac{1}{g}$	$-g$	$-\frac{1}{g}$	g
χ_{11}	3	$\frac{1}{a}$	$-d$	$-c$	d	c	$-i$	i		$\frac{1}{f}$		f	$-\frac{1}{f}$	$-f$	$-\frac{1}{a}$	$-a$	$\frac{1}{d}$	$-\frac{1}{c}$		-3	a	$-\frac{1}{d}$		$\frac{1}{c}$	$\frac{1}{g}$	$-g$	$-\frac{1}{g}$	g
χ_{12}	3	$\frac{1}{a}$	$-c$	$-d$	c	d	$-i$	i		$-f$		$-\frac{1}{f}$	f	$\frac{1}{f}$	$-\frac{1}{a}$	$-a$	$\frac{1}{c}$	$-\frac{1}{d}$		-3	a	$-\frac{1}{c}$		$\frac{1}{d}$	g	$-\frac{1}{g}$	$-g$	$\frac{1}{g}$
χ_{13}	3	a	$-\frac{1}{c}$	$-\frac{1}{d}$	$\frac{1}{c}$	$\frac{1}{d}$	i	$-i$		$-\frac{1}{f}$		$-f$	$\frac{1}{f}$	f	$-a$	$-\frac{1}{a}$	c	$-d$		-3	$\frac{1}{a}$	$-c$		d	$\frac{1}{g}$	$-g$	$-\frac{1}{g}$	g
χ_{14}	3	a	$-\frac{1}{d}$	$-\frac{1}{c}$	$\frac{1}{d}$	$\frac{1}{c}$	i	$-i$		f		$\frac{1}{f}$	$-f$	$-\frac{1}{f}$	$-a$	$-\frac{1}{a}$	d	$-c$		-3	$\frac{1}{a}$	$-d$		c	g	$-\frac{1}{g}$	$-g$	$\frac{1}{g}$
χ_{15}	3	$-a$	$\frac{1}{d}$	$\frac{1}{c}$	$-\frac{1}{d}$	$-\frac{1}{c}$	i	$-i$		f		$\frac{1}{f}$	$-f$	$-\frac{1}{f}$	a	$\frac{1}{a}$	$-d$	c		-3	$-\frac{1}{a}$	d		$-c$	g	$-\frac{1}{g}$	$-g$	$\frac{1}{g}$
χ_{16}	3	$-a$	$\frac{1}{c}$	$\frac{1}{d}$	$-\frac{1}{c}$	$-\frac{1}{d}$	i	$-i$		$-\frac{1}{f}$		$-f$	$\frac{1}{f}$	f	a	$\frac{1}{a}$	$-c$	d		-3	$-\frac{1}{a}$	c		$-d$	$\frac{1}{g}$	$-g$	$-\frac{1}{g}$	g
χ_{17}	3	-1	e	$\frac{1}{e}$	e	$\frac{1}{e}$	-1	-1		$\frac{1}{e}$		e	$\frac{1}{e}$	e	-1	-1	$\frac{1}{e}$	e		3	-1	$\frac{1}{e}$		e	$-g$	$-\frac{1}{g}$	$-g$	$-\frac{1}{g}$
χ_{18}	3	-1	$\frac{1}{e}$	e	$\frac{1}{e}$	e	-1	-1		e		$\frac{1}{e}$	e	$\frac{1}{e}$	-1	-1	e	$\frac{1}{e}$		3	-1	e		$\frac{1}{e}$	$-\frac{1}{g}$	$-g$	$-\frac{1}{g}$	$-g$
χ_{19}	3	1	$-\frac{1}{e}$	$-e$	$-\frac{1}{e}$	$-e$	-1	-1		e		$\frac{1}{e}$	e	$\frac{1}{e}$	1	1	$-e$	$-\frac{1}{e}$		3	1	$-e$		$-\frac{1}{e}$	$-\frac{1}{g}$	$-g$	$-\frac{1}{g}$	$-g$
χ_{20}	3	1	$-e$	$-\frac{1}{e}$	$-e$	$-\frac{1}{e}$	-1	-1		$\frac{1}{e}$		e	$\frac{1}{e}$	e	1	1	$-\frac{1}{e}$	$-e$		3	1	$-\frac{1}{e}$		$-e$	$-g$	$-\frac{1}{g}$	$-g$	$-\frac{1}{g}$
χ_{21}	3	$-i$	f	$-\frac{1}{f}$	f	$-\frac{1}{f}$	1	1		$-e$		$-\frac{1}{e}$	$-e$	$-\frac{1}{e}$	$-i$	i	$\frac{1}{f}$	$-f$		3	i	$\frac{1}{f}$		$-f$	$-\frac{1}{g}$	$-g$	$-\frac{1}{g}$	$-g$
χ_{22}	3	$-i$	$-\frac{1}{f}$	f	$-\frac{1}{f}$	f	1	1		$-\frac{1}{e}$		$-e$	$-\frac{1}{e}$	$-e$	$-i$	i	$-f$	$\frac{1}{f}$		3	i	$-f$		$\frac{1}{f}$	$-g$	$-\frac{1}{g}$	$-g$	$-\frac{1}{g}$
χ_{23}	3	i	$\frac{1}{f}$	$-f$	$\frac{1}{f}$	$-f$	1	1		$-\frac{1}{e}$		$-e$	$-\frac{1}{e}$	$-e$	i	$-i$	f	$-\frac{1}{f}$		3	$-i$	f		$-\frac{1}{f}$	$-g$	$-\frac{1}{g}$	$-g$	$-\frac{1}{g}$
χ_{24}	3	i	$-f$	$\frac{1}{f}$	$-f$	$\frac{1}{f}$	1	1		$-e$		$-\frac{1}{e}$	$-e$	$-\frac{1}{e}$	i	$-i$	$-\frac{1}{f}$	f		3	$-i$	$-\frac{1}{f}$		f	$-\frac{1}{g}$	$-g$	$-\frac{1}{g}$	$-g$
χ_{25}	4								-2		2								-1	-4			1		-4	4	4	-4
χ_{26}	4								1		-1								2	-4			-2		-4	4	4	-4
χ_{27}	4								-2		-2								1	4			1		4	4	4	4
χ_{28}	4								1		1								-2	4			-2		4	4	4	4

FIGURE 26. For $G = SL_2(\mathbb{F}_9)$, `Display(CharacterTable(Btilde));`, for $a = \frac{-1+i}{\sqrt{2}}, c = e^{7\pi i/12}, d = -e^{11\pi i/12}, e = \frac{1+i\sqrt{3}}{2}, f = -e^{\pi i/6}, f = \frac{3-3i\sqrt{3}}{2}$.

	$\|G\|$	18	18	$\|G\|$	18	18	10	10	10	10	8	8	8
	1	6	6	2	3	3	5	5	10	10	8	4	8
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	4	-1	2	-4	1	-2	-1	-1	1	1			
χ_3	4	2	-1	-4	-2	1	-1	-1	1	1			
χ_4	5	-1	2	5	-1	2					-1	1	-1
χ_5	5	2	-1	5	2	-1					-1	1	-1
χ_6	8	1	1	-8	-1	-1	a	$\overline{a}$	$-a$	$-\overline{a}$			
χ_7	8	1	1	-8	-1	-1	$\overline{a}$	a	$-\overline{a}$	$-a$			
χ_8	8	-1	-1	8	-1	-1	a	$\overline{a}$	a	$\overline{a}$			
χ_9	8	-1	-1	8	-1	-1	$\overline{a}$	a	$\overline{a}$	a			
χ_{10}	9			9			-1	-1	-1	-1	1	1	1
χ_{11}	10	1	1	10	1	1						-2	
χ_{12}	10	-1	-1	-10	1	1					$\sqrt{2}$		$-\sqrt{2}$
χ_{13}	10	-1	-1	-10	1	1					$-\sqrt{2}$		$\sqrt{2}$

FIGURE 27. For $G = SL_2(\mathbb{F}_9)$, `Display(CharacterTable(G));`, for $a = \frac{1+\sqrt{5}}{2}$.

	$\|\widetilde{G}\|$	$\|\widetilde{G}\|$	$\|\widetilde{G}\|$	$\|\widetilde{G}\|$	$\|\widetilde{G}\|$	$\|\widetilde{G}\|$	24	24	24	18	18	18	18	24	24	24	24	24	24	30	30	30	30	30	30	30	30	30	30	30	30
	1	2	3	6	3	6	12	12	4	3	6	3	6	24	24	24	24	8	8	30	15	10	5	30	15	10	5	30	15	30	15
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	a	a	$\frac{1}{a}$	$\frac{1}{a}$	f	$\frac{1}{f}$	-1					$-f$	$-f$	$-\frac{1}{f}$	$-\frac{1}{f}$	1	1	j	j	l	l	$\frac{1}{j}$	$\frac{1}{j}$	$\bar{l}$	$\bar{l}$	$\frac{1}{k}$	$\frac{1}{k}$	k	k
χ_3	3	3	a	a	$\frac{1}{a}$	$\frac{1}{a}$	f	$\frac{1}{f}$	-1					$-f$	$-f$	$-\frac{1}{f}$	$-\frac{1}{f}$	1	1	k	k	$\bar{l}$	$\bar{l}$	$\frac{1}{k}$	$\frac{1}{k}$	l	l	$\frac{1}{j}$	$\frac{1}{j}$	j	j
χ_4	3	3	$\frac{1}{a}$	$\frac{1}{a}$	a	a	$\frac{1}{f}$	f	-1					$-\frac{1}{f}$	$-\frac{1}{f}$	$-f$	$-f$	1	1	$\frac{1}{j}$	$\frac{1}{j}$	l	l	j	j	$\bar{l}$	$\bar{l}$	k	k	$\frac{1}{k}$	$\frac{1}{k}$
χ_5	3	3	$\frac{1}{a}$	$\frac{1}{a}$	a	a	$\frac{1}{f}$	f	-1					$-\frac{1}{f}$	$-\frac{1}{f}$	$-f$	$-f$	1	1	$\frac{1}{k}$	$\frac{1}{k}$	$\bar{l}$	$\bar{l}$	k	k	l	l	j	j	$\frac{1}{j}$	$\frac{1}{j}$
χ_6	4	-4	4	-4	4	-4				-2	2	1	-1							1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_7	4	-4	4	-4	4	-4				1	-1	-2	2							1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_8	5	5	5	5	5	5	1	1	1	-1	-1	2	2	-1	-1	-1	-1	-1	-1												
χ_9	5	5	5	5	5	5	1	1	1	2	2	-1	-1	-1	-1	-1	-1	-1	-1												
χ_{10}	6	6	b	b	$\frac{1}{b}$	$\frac{1}{b}$	g	$\frac{1}{g}$	2											$-\frac{1}{f}$	$-\frac{1}{f}$	1	1	$-f$	$-f$	1	1	$-f$	$-f$	$-\frac{1}{f}$	$-\frac{1}{f}$
χ_{11}	6	6	$\frac{1}{b}$	$\frac{1}{b}$	b	b	$\frac{1}{g}$	g	2											$-f$	$-f$	1	1	$-\frac{1}{f}$	$-\frac{1}{f}$	1	1	$-\frac{1}{f}$	$-\frac{1}{f}$	$-f$	$-f$
χ_{12}	6	-6	b	$-b$	$\frac{1}{b}$	$-\frac{1}{b}$								h	$-h$	$\frac{1}{h}$	$-\frac{1}{h}$	$-i$	i	$\frac{1}{f}$	$-\frac{1}{f}$	-1	1	f	$-f$	-1	1	f	$-f$	$\frac{1}{f}$	$-\frac{1}{f}$
χ_{13}	6	-6	b	$-b$	$\frac{1}{b}$	$-\frac{1}{b}$								$-h$	h	$-\frac{1}{h}$	$\frac{1}{h}$	i	$-i$	$\frac{1}{f}$	$-\frac{1}{f}$	-1	1	f	$-f$	-1	1	f	$-f$	$\frac{1}{f}$	$-\frac{1}{f}$
χ_{14}	6	-6	$\frac{1}{b}$	$-\frac{1}{b}$	b	$-b$								$-\frac{1}{h}$	$\frac{1}{h}$	$-h$	h	i	$-i$	f	$-f$	-1	1	$\frac{1}{f}$	$-\frac{1}{f}$	-1	1	$\frac{1}{f}$	$-\frac{1}{f}$	f	$-f$
χ_{15}	6	-6	$\frac{1}{b}$	$-\frac{1}{b}$	b	$-b$								$\frac{1}{h}$	$-\frac{1}{h}$	h	$-h$	$-i$	i	f	$-f$	-1	1	$\frac{1}{f}$	$-\frac{1}{f}$	-1	1	$\frac{1}{f}$	$-\frac{1}{f}$	f	$-f$
χ_{16}	8	8	8	8	8	8				-1	-1	-1	-1							l	l	l	l	l	l	$\bar{l}$	$\bar{l}$	$\bar{l}$	$\bar{l}$	$\bar{l}$	$\bar{l}$
χ_{17}	8	8	8	8	8	8				-1	-1	-1	-1							$\bar{l}$	$\bar{l}$	$\bar{l}$	$\bar{l}$	$\bar{l}$	$\bar{l}$	l	l	l	l	l	l
χ_{18}	8	-8	8	-8	8	-8				-1	1	-1	1							$-l$	l	$-l$	l	$-l$	l	$-\bar{l}$	$\bar{l}$	$-\bar{l}$	$\bar{l}$	$-\bar{l}$	$\bar{l}$
χ_{19}	8	-8	8	-8	8	-8				-1	1	-1	1							$-\bar{l}$	$\bar{l}$	$-\bar{l}$	$\bar{l}$	$-\bar{l}$	$\bar{l}$	$-l$	l	$-l$	l	$-l$	l
χ_{20}	9	9	9	9	9	9	1	1	1					1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_{21}	9	9	c	c	$\frac{1}{c}$	$\frac{1}{c}$	$-f$	$-\frac{1}{f}$	1					$-f$	$-f$	$-\frac{1}{f}$	$-\frac{1}{f}$	1	1	$\frac{1}{f}$	$\frac{1}{f}$	-1	-1	f	f	-1	-1	f	f	$\frac{1}{f}$	$\frac{1}{f}$
χ_{22}	9	9	$\frac{1}{c}$	$\frac{1}{c}$	c	c	$-\frac{1}{f}$	$-f$	1					$-\frac{1}{f}$	$-\frac{1}{f}$	$-f$	$-f$	1	1	f	f	-1	-1	$\frac{1}{f}$	$\frac{1}{f}$	-1	-1	$\frac{1}{f}$	$\frac{1}{f}$	f	f
χ_{23}	10	10	10	10	10	10	-2	-2	-2	1	1	1	1																		
χ_{24}	10	-10	10	-10	10	-10				1	-1	1	-1	i	$-i$	i	$-i$	i	$-i$												
χ_{25}	10	-10	10	-10	10	-10				1	-1	1	-1	$-i$	i	$-i$	i	$-i$	i												
χ_{26}	12	-12	d	$-d$	$\frac{1}{d}$	$-\frac{1}{d}$														j	$-j$	l	$-l$	$\frac{1}{j}$	$-\frac{1}{j}$	$\bar{l}$	$-\bar{l}$	$\frac{1}{k}$	$-\frac{1}{k}$	k	$-k$
χ_{27}	12	-12	d	$-d$	$\frac{1}{d}$	$-\frac{1}{d}$														k	$-k$	$\bar{l}$	$-\bar{l}$	$\frac{1}{k}$	$-\frac{1}{k}$	l	$-l$	$\frac{1}{j}$	$-\frac{1}{j}$	j	$-j$
χ_{28}	12	-12	$\frac{1}{d}$	$-\frac{1}{d}$	d	$-d$														$\frac{1}{j}$	$-\frac{1}{j}$	l	$-l$	j	$-j$	$\bar{l}$	$-\bar{l}$	k	$-k$	$\frac{1}{k}$	$-\frac{1}{k}$
χ_{29}	12	-12	$\frac{1}{d}$	$-\frac{1}{d}$	d	$-d$														$\frac{1}{k}$	$-\frac{1}{k}$	$\bar{l}$	$-\bar{l}$	k	$-k$	l	$-l$	j	$-j$	$\frac{1}{j}$	$-\frac{1}{j}$
χ_{30}	15	15	e	e	$\frac{1}{e}$	$\frac{1}{e}$	f	$\frac{1}{f}$	-1					f	f	$\frac{1}{f}$	$\frac{1}{f}$	-1	-1												
χ_{31}	15	15	$\frac{1}{e}$	$\frac{1}{e}$	e	e	$\frac{1}{f}$	f	-1					$\frac{1}{f}$	$\frac{1}{f}$	f	f	-1	-1												

FIGURE 28. For $G = SL_2(\mathbb{F}_9)$, `Display(CharacterTable(Gtilde));`, for $a = -3\frac{1+i\sqrt{3}}{2}, b = 2a, c = 3a, d = 2b, e = 5a, f = a/3, g = 2a/3, h = -e^{\pi i/12} - e^{7\pi i/12}, i = -\sqrt{2}, j = -e^{4\pi i/15} + e^{\pi i/15}, k = -e^{22\pi i/15} - e^{28\pi i/15}, l = \frac{1-\sqrt{5}}{4}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
θ_1	1	1	1	1	1	1	1	1				
θ_2											1	1
θ_3											1	1
θ_4									1	1		
θ_5									1	1		
θ_6									1	1		
θ_7									1	1		
θ_8											1	1
θ_9											1	1

FIGURE 29. For $G = SL_2(\mathbb{F}_9)$, `charCoeffs(U, B);`.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}
θ_1	1									1			
θ_2				1	1								
θ_3												1	
θ_4													1
θ_5												1	
θ_6													1
θ_7											1		
θ_8											1		
θ_9			1			1	1					1	1
θ_{10}				1				1	1	1	1		
θ_{11}		1				1	1					1	1
θ_{12}					1			1	1	1	1		

FIGURE 30. For $G = SL_2(\mathbb{F}_9)$, `charCoeffs(B, G);`.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}	χ_{27}	χ_{28}
θ_1	1	1	1	1	1	1	1	1																				
θ_2																										1		1
θ_3																										1		1
θ_4																									1		1	
θ_5																									1		1	
θ_6																										1		1
θ_7																										1		1
θ_8																									1		1	
θ_9																									1		1	
θ_{10}									1			1		1	1		1			1		1	1					
θ_{11}										1	1		1			1		1	1		1			1				

FIGURE 31. For $G = SL_2(\mathbb{F}_9)$, `charCoeffs(Utilde, Btilde);`.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}	χ_{27}	χ_{28}	χ_{29}	χ_{30}	χ_{31}
θ_1	1																			1											
θ_2								1	1																						
θ_3																								1							
θ_4																									1						
θ_5																									1						
θ_6																								1							
θ_7																							1								
θ_8																							1								
θ_9												1														1	1				
θ_{10}															1													1	1		
θ_{11}														1														1	1		
θ_{12}													1													1	1				
θ_{13}														1														1	1		
θ_{14}													1													1	1				
θ_{15}												1														1	1				
θ_{16}															1													1	1		
θ_{17}																														2	
θ_{18}																															2
θ_{19}				1	1																	1									1
θ_{20}		1	1																		1									1	
θ_{21}											1											1									1
θ_{22}										1											1									1	
θ_{23}										1											1									1	
θ_{24}											1											1									1
θ_{25}						1												1	1					1	1						
θ_{26}							1											1	1					1	1						
θ_{27}								1								1	1			1			1								
θ_{28}									1							1	1			1			1								

FIGURE 32. For $G = SL_2(\mathbb{F}_9)$, `charCoeffs(Btilde, Gtilde);`.

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