THE MIRROR SYMMETRY CONJECTURE FOR NETWORKS ON SURFACES

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1. Measurements on Networks

Many of the basic definitions are the same as those in [2]. We reproduce them here for the sake of convenience.

1.1. **Networks on surfaces.** Let S be a compact connected oriented surface, possibly with boundary. A *simple-crossing, vertex weighted oriented network on* S is a directed graph $\mathcal{N} = (V, E)$ embedded on S that satisfies the *simple-crossing* condition: $V = B \sqcup I$, where B is the set of *boundary vertices* (each embedded on ∂S) and I is the set of *interior vertices* (each embedded on $\operatorname{int} S$), each boundary vertex is a source or a sink (i.e., has in-degree 0 and out-degree 1, or vice-versa), and each interior vertex has in-degree 2 and out-degree 2, and the incoming edges (resp. outgoing edges) are adjacent. Furthermore, each interior vertex $v \in I$ is assigned an indeterminate weight x_v. Throughout the following, we will refer to $\mathcal{N}$ simply as a *network* on S.

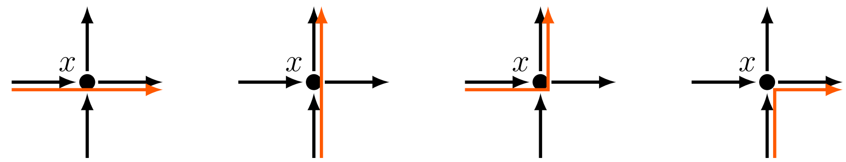
Fixing a network $\mathcal{N}$ on a surface S, we study paths and walks on $\mathcal{N}$, which come in several types. By a *(boundary) path* we mean a directed walk on $\mathcal{N}$ from some $v \in B$ to some $w \in B$ (so v is a source vertex and w is a sink) that may possibly traverse interior vertices and edges more than once. We use the term *closed walk* in the usual way (so it also may possibly self-intersect in vertices and edges). The collections of all boundary paths and closed walks on $\mathcal{N}$ are denoted by $\mathcal{P}(\mathcal{N})$ and $\mathcal{C}(\mathcal{N})$, respectively. Later we also consider *flows*, which we define similarly to Talaska's definition: a flow is a collection $F = P \sqcup C$ where $P \subset \mathcal{P}(\mathcal{N})$, $C \subseteq \mathcal{C}(\mathcal{N})$, each element $f \in F$ is self-avoiding, and no two $f, f' \in F$ share a common edge.[1] We call an element $f \in F$ a *component of* F, and we denote the collection of all flows on $\mathcal{N}$ by $\mathcal{F}(\mathcal{N})$. Later it will be useful to consider those flows whose components are only cycles, no boundary paths (resp. only boundary paths, no cycles); in those cases we will refer to them explicitly as *cycle flows* (resp. *path flows*).

1.2. **Measurements of classes.** Now let p be any boundary path, closed walk, or component of a flow on $\mathcal{N}$; we consider two different ways of assigning a weight to p that we call the *highway* and *underway measurements* of p. For both, the setup is the same: at each interior vertex $v \in I$ that p traverses, p accumulates a certain weight contribution that depends on how p traverses v. The total weight of p is defined to be the product of these weight contributions over all interior vertices (with multiplicities). There are four possible local pictures for how p may traverse v; Figure 1 below lists the highway/underway weight contributions in each case.

We denote the highway and underway measurements of p by $\mathrm{wt}_H(p)$ and $\mathrm{wt}_U(p)$, respectively. Note that at most one of $\mathrm{wt}_H, \mathrm{wt}_U$ is nonzero for any given p. Thus we call p a *highway path* (resp. *underway path*) if $\mathrm{wt}_H(p) \neq 0$ (resp. $\mathrm{wt}_U(p) \neq 0$).

Now that we have defined a way of assigning a weight to any specific path or walk on $\mathcal{N}$, we wish to group these measurements in ways that are meaningful with respect to the topological data of

[1]Note here that we require that two components $f, f' \in F$ be only edge-disjoint, while Talaska requires vertex-disjointedness. This is because we consider vertex-weighted networks, while she considers edge-weighted networks. Later, when we reinterpret our vertex-weighted networks as edge-weighted graphs, our definition of flow coincides with hers.



contribution factor for:				
highway weight:	x	0	1	1
underway weight:	0	x	1	1

FIGURE 1. The weight contribution factors for highway/underway measurements for each of the possible ways a path (shown in orange) may traverse a vertex (that has weight x).

the surface S. To this end, let $H = H_1(S, \mathbb{Z})$ be the first homology group of S, and let $\mathcal{C}(\mathcal{N})$ and $\mathcal{P}(\mathcal{N})$ be the collections of closed walks and *boundary paths* (i.e., walks from sources to sinks) on $\mathcal{N}$, respectively. Note that a given closed walk $c \in \mathcal{C}(\mathcal{N})$ defines a homology class $[c] \in H$. If two paths $p, q \in \mathcal{P}(\mathcal{N})$ start at the same source and end at the same sink, then we say that p and q are *homologous*, written $p \sim q$, if the closed walk $p \cup q^* \in \mathcal{C}(\mathcal{N})$ has trivial homology class (here, q^* denotes the path on S obtained by following q in reverse).

Now given a boundary path $p \in \mathcal{P}(\mathcal{N})$, we call the quantity

$$M_H^{[p]} = \sum_{p' : p' \sim p} \mathrm{wt}_H(p')$$

the *highway measurement* of the class $[p]$. Throughout the following, we will define everything in terms of highway measurements for simplicity's sake; the corresponding definitions for underway measurements are made completely analogously, simply by replacing each subscript H with a U.

We wish to make a similar definition for $M_H^{[c]}$ given a closed walk $c \in \mathcal{C}(\mathcal{N})$, but we must make a slight adjustment. We define the *multiplicity* $\mathrm{mult}(c)$ *of* c to be the maximum $k \in \mathbb{N}$ such that c is obtained by repeating some other (shorter) closed walk $c' \in \mathcal{C}(\mathcal{N})$. Then we define

$$M_H^{[c]} = \sum_{c' : [c'] = [c]} \frac{1}{\mathrm{mult}(c')} \mathrm{wt}_H(c'),$$

the *highway measurement* of the class $[c]$.

Next we define measurements of flows and how to group them. Given a flow $F \in \mathcal{F}(\mathcal{N})$, we define its highway weight $\mathrm{wt}_H(F)$ to be the product of each weight $\mathrm{wt}_H(f)$ of a constitutent $f \in F$. Indeed, a flow F also defines a homology class $[F] \in H$: because H is commutative, it makes sense to define $[F] \coloneqq \prod_{f \in F} [f]$. However, it turns out to be more useful to endow $\mathcal{F}(\mathcal{N})$ with a finer equivalence relation. In particular, for $F, F' \in \mathcal{F}(\mathcal{N})$, we say $F \sim F'$ if F and F' correspond the same homology class in H and they are supported on the same sets of source and sink vertices. By abuse of notation, from now on we will write $[F]$ to mean the $\sim$-equivalence class of $F \in \mathcal{F}(\mathcal{N})$.

As in the case of closed walks, we need to make a slight (different) adjustment to properly define $M_H^{[F]}$. To do this, note that a flow $F \in \mathcal{F}(\mathcal{N})$ induces a bijection $\phi_F : B^+ \to B^-$ between $B^+, B^- \subset B$, the sets of sinks and sources of $\mathcal{N}$. If we fix orderings on B^+ and B^-,[2] we can associate ϕ_F (and hence F) to a permutation $\sigma_F \in S_{|B|/2}$. Recalling that F has a unique decomposition

[2]In practice one needs only to fix an ordering on B^+, and then there are canonical ways to have this ordering induce an ordering on B^-.

$F = P \sqcup C$, where $P \subseteq \mathcal{P}(\mathcal{N})$ and $C \subseteq \mathcal{C}(\mathcal{N})$, we define the sign $\operatorname{sgn}(F)$ of the flow F to be

$$\operatorname{sgn}(F) = \operatorname{sgn}(\sigma_F) \cdot (-1)^{|C|}.$$

Finally, we then define

$$M_H^{[F]} = \sum_{F' : [F'] = [F]} \operatorname{sgn}(F') \operatorname{wt}_H(F'),$$

the *highway measurement* of the class $[F]$.

1.3. **The mirror symmetry conjecture.** Now that we have developed meaningful ways of measuring classes of paths, closed walks, and flows on $\mathcal{N}$, we come to our main objects of interest: rings generated by class measurements.

Definition 1.1. Given a network $\mathcal{N}$, we define the **type I highway measurement ring associated to $\mathcal{N}$** as

$$R_H^{\mathrm{I}}(\mathcal{N}) = \langle M_H^{[p]}, M_H^{[c]} \mid p \in \mathcal{P}(\mathcal{N}), c \in \mathcal{C}(\mathcal{N}) \rangle_{\mathbb{Q}},$$

where $\langle \cdot \rangle_{\mathbb{Q}}$ denotes ring generation over $\mathbb{Q}$. Furthermore, we define the **type II highway measurement ring associated to $\mathcal{N}$** as

$$R_H^{\mathrm{II}}(\mathcal{N}) = \langle M_H^{[F]} \mid F \in \mathcal{F}(\mathcal{N}) \rangle_{\mathbb{Q}}.$$

For each of the two previous definitions, we define the corresponding underway rings $R_U^{\mathrm{I}}(\mathcal{N}), R_U^{\mathrm{II}}(\mathcal{N})$ in the obvious ways.

Now we are ready to state the **mirror symmetry conjecture** for networks on surfaces, the main subject of this report.

Conjecture 1.2 (Lam-Pylyavskyy)**.** *Let $\mathcal{N}$ be any network. Then*

$$R_H^{\mathrm{I}}(\mathcal{N}) = R_U^{\mathrm{I}}(\mathcal{N}) = R_H^{\mathrm{II}}(\mathcal{N}) = R_U^{\mathrm{II}}(\mathcal{N}).$$

2. Relating Type I and Type II Measurements

The first step in proving Conjecture 1.2 is to prove that the type I and type II highway measurements generate the same ring, and similarly that type I and type II underway measurements do the same. In other words, we would like to show that $R_H^{\mathrm{I}}(\mathcal{N}) = R_H^{\mathrm{II}}(\mathcal{N})$ and $R_U^{\mathrm{I}}(\mathcal{N}) = R_U^{\mathrm{II}}(\mathcal{N})$.

At the present, we are not able to show these equalities for a general network on a surface; we are able, however, to prove them for a specific important class of networks on tori: the so-called **(n, m, k)-torus** networks, as described in [1]. That is, we will work towards and then prove the following result in this section.

Theorem 2.1. *If $\mathcal{N}$ is a (n, m, k)-torus, then $R_H^{\mathrm{I}}(\mathcal{N}) = R_H^{\mathrm{II}}(\mathcal{N})$, and $R_U^{\mathrm{I}}(\mathcal{N}) = R_U^{\mathrm{II}}(\mathcal{N})$.*

We will begin by demonstrating how to construct two edge-weighted graphs from a given network—one which corresponds to highway measurements and one which corresponds to underway measurements—and we will define and relate type I and type II measurements on such graphs. In the second subsection, we will show how to encode the homological data of a network in the edge-weighted graph, and we will hence deduce algebraic relations between type I and type II measurements on an (n, m, k)-torus.

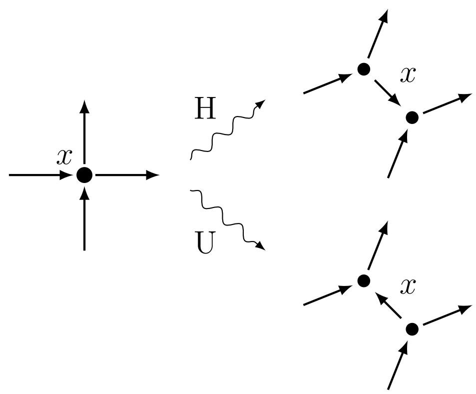


FIGURE 2. The local transformation of $\mathcal{N}$ to $G_H(\mathcal{N})$ (above) and $G_U(\mathcal{N})$ (below). Unmarked edges have weight 1.

2.1. **Edge-weighted graphs.** Given any (vertex-weighted) network $\mathcal{N}$, we want to be able to describe measurements on $\mathcal{N}$ as measurements on an edge-weighted graph, for which there are specific tools—like adjacency matrices—that facilitate measurement computation. It turns out that this is most easily accomplished by constructing two separate graphs: $G_H(\mathcal{N})$, which encodes highway measurements, and $G_U(\mathcal{N})$, which encodes underway measurements.

The graphs $G_H(\mathcal{N})$ and $G_U(\mathcal{N})$ are constructed by changing $\mathcal{N}$ locally at each interior vertex as shown in Figure 2.1. Note that any path through a vertex in $\mathcal{N}$ preserves its highway and underway measurements in the corresponding paths in $G_H(\mathcal{N})$ and $G_U(\mathcal{N})$, respectively. This justifies our approach of considering measurements on edge-weighted graphs.

Now we wish to introduce type I and type II measurements for edge-weighted graphs in such a way that they agree with the corresponding measurements of a network $\mathcal{N}$ when applied to $G_H(\mathcal{N})$ and $G_U(\mathcal{N})$. This is best accomplished by encoding the measurements in generating functions.

Given an arbitrary edge-weighted directed graph G, let $\mathcal{C}(G)$ be the set of non-empty closed walks in G, and let $\mathcal{F}(G)$ be the set of flows (collections of non-intersecting[3] simple cycles) on G. Then we define the **type I generating function** for G as

$$\mathrm{M}^{\mathrm{I}}_C(G) = \sum_{c\in\mathcal{C}(G)} \frac{\mathrm{wt}(c)}{\mathrm{mult}(c)},$$

where $\mathrm{wt}(c)$ is the product of the weights of the edges in the closed walk, and $\mathrm{mult}(c)$ is defined as it was for closed walks on networks. We also define the **type II generating function** for G as

$$\mathrm{M}^{\mathrm{II}}_C(G) = \sum_{F\in\mathcal{F}(G)} (-1)^{|F|}\,\mathrm{wt}(F).$$

Remark 2.2. We use the subscript C in these generating functions to emphasize that these measurements are only taken over closed walks in G. Later we will define more general measurements on G using a set of distinguished "boundary vertices" as well.

Note that these measurements agree with those defined on networks when restricting to closed walks in the network. In other words,

$$\mathrm{M}^{\mathrm{I}}_C(G_H(\mathcal{N})) = \sum_{[c]} M_H^{[c]}, \tag{1}$$

[3]Here, we take "non-intersecting" to mean that no two cycles in a given flow share a common vertex. This is to be contrasted with the corresponding definition for networks, where flow components may share a common vertex, but not a common edge.

where the sum is taken over all equivalence classes of closed walks $[c]$ on $\mathcal{N}$, and

$$\mathrm{M}_C^{\mathrm{II}}(G_H(\mathcal{N})) = \sum_{[F]} M_H^{[F]}, \tag{2}$$

where the sum is taken over all equivalence classes $[F]$ of cycle flows on $\mathcal{N}$. Similar statements are true for $G_U(\mathcal{N})$ and underway measurements.

We now prove two lemmas that together imply algebraic relations between type I and type II measurements.

Lemma 2.3. *Let G be an edge-weighted directed graph, and let A be its adjacency matrix. Then*

$$\det(I - A) = \mathrm{M}_C^{\mathrm{II}}(G).$$

Proof. Suppose that G has n vertices, and let $a_{i,j}$ be the i,j-th entry of $I - A$. Recall the permutation definition of the determinant:

$$\det(I - A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

We can associate each term $\operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}$ of this sum to a cycle flow in which each cycle is of the form $i \to \sigma(i) \to \sigma^2(i) \to \cdots \to i$. The product $\prod_{i=1}^{n} a_{i,\sigma(i)}$ gives us the weight of the flow defined by σ up to some sign.

Because we are taking the determinant of $I - A$ (as opposed to simply $-A$), some of the terms in the expansion of $\prod_{i=1}^{n} a_{i,\sigma(i)}$ will have a weight contribution of 1 from the loop $i \to i$. This corresponds to taking cycle flows in which not every vertex is a member of some cycle.

The sign of each term in the expansion of $\operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}$ is therefore $(-1)^{k+\ell}$, where $\operatorname{sgn}(\sigma) = (-1)^{\ell}$ and k is the number of vertices of G used in the cycle flow corresponding to this term. Suppose σ has cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. Then $k = k' + \sum_{\lambda_i > 1} \lambda_i$, where k' is the number of loops in this family, and also $\ell \equiv \sum_{\lambda_i > 1}(\lambda_i - 1) \pmod 2$. Hence

$$k + \ell \equiv k' + \sum_{\lambda_i > 1} \lambda_i + \sum_{\lambda_i > 1} (\lambda_i - 1) \equiv k' + \sum_{\lambda_i > 1} 1,$$

which is exactly the number of cycles in the corresponding flow. Because permutations are a bijection, each vertex of G is used at most once, so the corresponding flows are non-intersecting. Therefore, because every cycle flow appears in the determinant exactly once,

$$\det(I - A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{F \in \mathcal{F}(G)} (-1)^{|F|} \operatorname{wt}(F) = \mathrm{M}_C^{\mathrm{II}}(G).$$

□

Lemma 2.4. *Let G be an edge-weighted directed graph, and let A be its adjacency matrix. Then*

$$-\operatorname{tr}(\log(I - A)) = \mathrm{M}_C^{\mathrm{I}}(G).$$

Proof. For any $k \in \mathbb{N}$, notice that $\operatorname{tr}(A^k)$ gives the sum of weights of all k-step cycles in G, where cycles with different base points are counted independently. Any k-step cycle starting at some vertex i_1 can be written uniquely as $(i_1\ i_2\ \cdots\ i_r)^s$, where the corresponding cycle is $i_1 \to i_2 \to \cdots \to i_r \to i_1 \to \cdots$, $rs = k$, and $(i_1\ i_2\ \cdots\ i_r)$ is not a power of some smaller cycle.

Notice, however, that when disregarding base point, $(i_1\ i_2\ \cdots\ i_r)^s = (i_2\ \cdots\ i_r\ i_1)^s = \cdots$, so in the sum $\operatorname{tr}(A^k)$ the weight of the cycle will be overcounted exactly r times. Dividing by k, the weight of the cycle in $\operatorname{tr}(A^k)/k$ will be

$$\frac{r}{k} \cdot \operatorname{wt}(i_1\ \cdots\ i_r)^s = \frac{\operatorname{wt}(i_1\ \cdots\ i_r)^s}{s}.$$

Since s is the multiplicity of this cycle, this shows that

$$\frac{\operatorname{tr}(A^k)}{k} = \sum_{p \in C_k(G)} \frac{\operatorname{wt}(p)}{\operatorname{mult}(p)},$$

where $C_k(G)$ is the set of all k-step closed walks in G (without regard to base points). Hence

$$\sum_{k=1}^{\infty} \frac{\operatorname{tr}(A^k)}{k} = \sum_{p \in C(G)} \frac{\operatorname{wt}(p)}{\operatorname{mult}(p)} = \mathrm{M}_C^{\mathrm{I}}(G).$$

Since $\log(I - A) = -\sum_{k=1}^{\infty} \frac{A^k}{k}$, we have

$$-\operatorname{tr} \circ \log(I - A) = \operatorname{tr}(-\log(I - A)) = \operatorname{tr}\left(\sum_{k=1}^{\infty} \frac{A^k}{k}\right) = \sum_{k=1}^{\infty} \frac{\operatorname{tr}(A^k)}{k} = \mathrm{M}_C^{\mathrm{I}}(G).$$

□

From two results we deduce our first important relation between type I and type II measurements.

Theorem 2.5. *Let G be an edge-weighted directed graph. Then*

$$\mathrm{M}_C^{\mathrm{I}}(G) = -\log(\mathrm{M}_C^{\mathrm{II}}(G)).$$

Proof. Apply Lemmas 2.3 and 2.4, as well as the well-known identity $\operatorname{tr} \circ \log = \log \circ \det$. □

2.2. **Application to toric networks.** Applying Theorem 2.5 to the edge-weighted graph derived from a network is almost enough to deduce algebraic relations between type I and type II measurements on that network. However, we must modify the generating functions we defined in the previous subsection so that they group measurements by homology class, as was specified in the first section.

Fix a network $\mathcal{N}$ on a surface S, and let $H = H_1(S, \mathbb{Z})$ be its first homology group. In particular, fix an identification polygon P for S, and let $g_1, \dots, g_a$ be the generators of H corresponding to the edges of P (so P is a $2a$-gon). Fix orientations for the g_i, and there is an induced isomorphism $\mathbf{h} : H \to \mathbb{Z}^a$ that maps a directed subgraph D of $\mathcal{N}$ to a vector $\mathbf{h}(D) = (h_1, \dots, h_a)$, where h_i is the number of edges of D that cross the generator g_i in the positive direction minus the number that cross g_i in the negative direction. Because $\mathbf{h}$ is well-defined on equivalence classes of paths (resp. closed walks, cycles, flows), then we will write $\mathbf{h}[p]$ for the image of the path (resp. closed walk, cycle, flow) class $[p]$ under $\mathbf{h}$.

Next we introduce generating functions for network measurements that are decorated according to homology type. These will be Laurent series[4] in the set of variables $\mathbf{t} = (t_1, \dots, t_a)$, and if $\mathbf{i} = (i_1, \dots, i_a) \in \mathbb{Z}^a$, then we will use the notation $\mathbf{t}^{\mathbf{i}}$ for the monomial $t_1^{i_1} \cdots t_a^{i_a}$. Now we define the **(homology-)decorated type I highway measurement generating function** for $\mathcal{N}$ to be

$$\mathcal{M}_H^{\mathrm{I}}(\mathcal{N}, \mathbf{t}) = \sum_{[p]} M_H^{[p]} \mathbf{t}^{\mathbf{h}[p]},$$

where the sum is taken over all equivalence classes of boundary paths *and* closed walks $[p]$ on the network $\mathcal{N}$. We define all other possible variants (type II, underway, etc.) of this generating function completely analogously.

Now we want to modify the edge-weighted graphs $G_H(\mathcal{N})$ and $G_U(\mathcal{N})$ so that measurements on these graphs will be decorated in a way that is consistent with the above definition of the decorated generating function. In particular, let $\mathcal{G}_H(\mathcal{N}, \mathbf{t})$ be the graph $G_H(\mathcal{N})$ embedded on P with the

[4]It should be emphasized that they are not necessarily *formal* Laurent series; i.e., a priori they may have infinitely many nonzero terms of negative exponent.

following modification: for each i, the weight of each edge that traverses the generator g_i in the positive (resp. negative) direction accrues a factor of t_i (resp. t_i^{-1}). We define $\mathcal{G}_U(\mathcal{N}, \mathbf{t})$ completely analogously.

We would like to relate the decorated measurement generating functions to measurements on these decorated graphs, in analogy with equations (1) and (2) above. However, our constructions of generating functions for edge-weighted graphs only account for closed walk class measurments, while the decorated measurement functions we have just defined include path class measurements. In general, this is a subtle issue to resolve, but it is trivially handled in the case of (n, m, k)-torus networks, since there are no boundary paths on these networks.

Hence we will now focus on this type of network for the remainder of this section. Fix integers $n, m \geq 1, k \geq 0$, and let $\mathcal{N}$ be the (n, m, k)-torus. The next proposition gives the desired analogy of (1) for the network $\mathcal{N}$; its proof is simply the observation made in the proceeding paragraph.

Proposition 2.6. *Let $\mathcal{N}$ be the (n, m, k)-torus. Then*

$$\mathrm{M}_C^{\mathrm{I}}(\mathcal{G}_H(\mathcal{N}, \mathbf{t})) = \mathcal{M}_H^{\mathrm{I}}(\mathcal{N}, \mathbf{t}),$$

and similarly for the other types of measurements (type II, underway, etc.).

Earlier we noted that $\mathcal{M}_H^{\mathrm{I}}(\mathcal{N}, \mathbf{t})$ and $\mathcal{M}_H^{\mathrm{II}}(\mathcal{N}, \mathbf{t})$ are generally Laurent series in $\mathbf{t}$. However, things are again much simpler for the (n, m, k)-torus.

Proposition 2.7. *Let $\mathcal{N}$ be the (n, m, k)-torus. Then $\mathcal{M}_H^{\mathrm{I}}(\mathcal{N}, \mathbf{t})$ and $\mathcal{M}_H^{\mathrm{II}}(\mathcal{N}, \mathbf{t})$ are a formal power series and a polynomial in $\mathbf{t}$, respectively (and similarly for underway measurements).*

Proof. Let $\mathcal{N}$ be a (n, m, k)-torus. We can orient the generators of the first homology group of the torus so that every every subgraph of $\mathcal{N}$ crosses the generators only in the positive direction; hence $\mathcal{M}_H^{\mathrm{I}}(\mathcal{N}, \mathbf{t})$ and $\mathcal{M}_H^{\mathrm{II}}(\mathcal{N}, \mathbf{t})$ are formal power series in $\mathbf{t}$. In fact, the latter is a polynomial, since there are only finitely many classes of cycle flows on $\mathcal{N}$ with nonzero highway measurement. □

By applying Propositions 2.6 and 2.7 and Theorem 2.5, we deduce the following proposition.

Proposition 2.8. *Let $\mathcal{N}$ be the (n, m, k)-torus. Then we have*

$$\mathcal{M}_H^{\mathrm{I}}(\mathcal{N}, \mathbf{t}) = -\log(\mathcal{M}_H^{\mathrm{II}}(\mathcal{N}, \mathbf{t})), \tag{3}$$

and similarly for underway measurements.

Finally, this proposition allows us to prove Theorem 2.1.

Proof of Theorem 2.1. Equation (3) implies that each coefficient of $\mathcal{M}_H^{\mathrm{I}}(\mathcal{N}, \mathbf{t})$ is expressible as a finite algebraic combination of coefficients of $\mathcal{M}_H^{\mathrm{II}}(\mathcal{N}, \mathbf{t})$, and vice versa. But the coefficients of the former generate $R_H^{\mathrm{I}}(\mathcal{N})$, and the coefficients of the latter generate $R_H^{\mathrm{II}}(\mathcal{N})$. Therefore $R_H^{\mathrm{I}}(\mathcal{N}) = R_H^{\mathrm{II}}(\mathcal{N})$. The analogous argument shows that $R_U^{\mathrm{I}}(\mathcal{N}) = R_U^{\mathrm{II}}(\mathcal{N})$. □

3. Relating Type II Highway and Underway Measurements

Fix integers $n, m \geq 1, k \geq 0$, and let $\mathcal{N}$ be the (n, m, k)-torus. We showed in the previous section that $R_H^{\mathrm{I}}(\mathcal{N}) = R_H^{\mathrm{II}}(\mathcal{N})$ and $R_U^{\mathrm{I}}(\mathcal{N}) = R_U^{\mathrm{II}}(\mathcal{N})$. In this section we will show that $R_H^{\mathrm{II}}(\mathcal{N}) = R_U^{\mathrm{II}}(\mathcal{N})$, which will in turn prove the mirror symmetry conjecture for this specific network. We do this by constructing a weight-preserving bijection between the sets of highway flows and underway flows that also preserves the homological equivalence of flows. This will show that the two rings are generated by the same measurments, hence proving their equality.

3.1. **The flow complement.** Let $\mathcal{N}$ be an arbitrary network on a surface. The *flow complement* is an involution on $\mathcal{F}(\mathcal{N})$, the set of flows on $\mathcal{N}$. We define this involution by considering the local picture of a flow at a vertex of $\mathcal{N}$. If the vertex is an interior vertex, there are six possibilities:

(1) a single path may enter and exit via the highway;
(2) a single path may enter and exit via the underway;
(3) a single path may enter via the highway and exit via the underway;
(4) a single path may enter via the underway and exit via the highway;
(5) one path may enter via the highway and exit via the underway, while another path enters via the underway and exits via the highway;
(6) no path traverses the vertex.

If the vertex is on the boundary of $\mathcal{N}$, there are two possibilites:

(7) a single path exits (resp. enters) the source (resp. sink);
(8) no path exits (resp. enters) the source (resp. sink).

If $F \in \mathcal{F}(\mathcal{N})$, the *flow complement* of F, denoted F^{c}, is the flow whose local pictures are the "complements" of the local pictures of F, where we define the complementary pairs of local pictures to be: (1) and (2), (3) and (4), (5) and (6), and (7) and (8). It is a straightforward exercise to check that this transformation is a well-defined involution on $\mathcal{F}(\mathcal{N})$.

Example 3.1. Figure 3 shows a highway flow F on the $(3, 4, 0)$-torus and its complement. Note how F^{c} is a complement of F in $\mathcal{N}$ when we consider each object as a collection of edges.

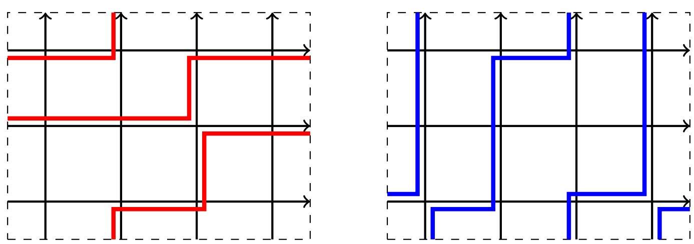


FIGURE 3. To the left is a highway flow F on the $(3, 4, 0)$-torus consisting of a single cycle. To the right is its complement.

Proposition 3.2. *If $F \in \mathcal{F}(\mathcal{N})$, then $\mathrm{wt}_H(F) = \mathrm{wt}_U(F^{\mathsf{c}})$. In particular, the family complement maps highway flows to underway flows and vice versa.*

Proof. Note that the flow complement is locally weight-preserving. Because a flow's weight is the product over the weights of its components, the flow complement is globally weight-preserving. ☐

Since we group flow measurements by homology class, the following propostion is essential.

Proposition 3.3. *Let $F_1, F_2 \in \mathcal{F}(\mathcal{N})$. If $F_1 \sim F_2$, then $F_1^{\mathsf{c}} \sim F_2^{\mathsf{c}}$.*

Proof. We use the setup and notation of subsection 2.2. Write $\mathbf{h}(\mathcal{N}) = (n_1, \dots, n_a) \in \mathbb{Z}^a$, and let $F \in \mathcal{F}(\mathcal{N})$ be an arbitrary flow supported on the sources $X \subseteq B^+$ and sinks $Y \subseteq B^-$. Since F^{c} contains precisely the edges of $\mathcal{N}$ not in F, we have $\mathbf{h}[F^{\mathsf{c}}] = \mathbf{h}(\mathcal{N}) - \mathbf{h}[F]$, and we see that F^{c} is supported on the sources $B^+ \setminus X$ and sinks $B^- \setminus Y$.

Now let $F_1, F_2 \in \mathcal{F}(\mathcal{N})$ be arbitrary flows such that $F_1 \sim F_2$, so $\mathbf{h}[F_1] = \mathbf{h}[F_2]$ and F_1, F_2 are supported on the same sources and sinks. From the above, it is easy to see that $\mathbf{h}[F_1^{\mathsf{c}}] = \mathbf{h}[F_2^{\mathsf{c}}]$ and that $F_1^{\mathsf{c}}, F_2^{\mathsf{c}}$ are supported on the same subsets of sources and sinks; hence $F_1^{\mathsf{c}} \sim F_2^{\mathsf{c}}$. ☐

To summarize, the flow complement is a weight-preserving bijection between the sets of highway flows and underway flows that also preserves the equivalence of flows.

3.2. **The mirror symmetry conjecture for the (n,m,k)-torus.** Now let $\mathcal{N}$ be the (n,m,k)-torus. Using the results of the previous subsection, we prove the following theorem.

Theorem 3.4. *If $\mathcal{N}$ is the (n,m,k)-torus, then $R_H^{\mathrm{II}}(\mathcal{N}) = R_U^{\mathrm{II}}(\mathcal{N})$.*

Proof. Let $F \in \mathcal{F}(\mathcal{N})$ be a flow, and let $\mathbf{h}[F] = (a,b) \in \mathbb{Z}^2$. By Theorem 7.2 of [1], we have $\operatorname{sgn}(F) = (-1)^{ab-a-b}$. In particular, if $F' \in \mathcal{F}(\mathcal{N})$ and $F' \sim F$, then $\operatorname{sgn}(F') = \operatorname{sgn}(F)$. Therefore, applying the results of the previous subsection, we see that the flow complement on $\mathcal{N}$ preserves flow class measurments; i.e., $M_H^{[F]} = M_H^{[F^c]}$. Thus the rings $R_H^{\mathrm{II}}(\mathcal{N})$ and $R_U^{\mathrm{II}}(\mathcal{N})$ are generated by precisely the same set of measurements, and hence they are equal. □

Putting together Theorems 2.1 and 3.4 now implies the Mirror Symmetry Conjecture for $\mathcal{N}$.

Theorem 3.5. *The mirror symmetry conjecture is true for any (n,m,k)-torus network.*

4. The Mirror Symmetry Conjecture for Other Networks

Although we have proven the mirror symmetry conjecture only for (n,m,k)-torus networks, many of the techniques we have developed in the previous sections can be applied in further generality. However, alone they are not enough, and as of yet we do not have a proof of the conjecture in full generality. In this section we approach a proof a general version of Theorem 2.1 by proposing the use of a theorem of Talaska, and we reduce the conjectural general version of Theorem 3.4 to a single, simpler conjecture.

4.1. **Generalizing Theorem 2.1.** Although Theorem 2.5 holds for any directed graph, it is not always meaningful to apply it to a homology-decorated graph as we did to prove Proposition 2.8. This specialization worked for the (n,m,k)-torus precisely because of Propositions 2.6 and 2.7; i.e., the torus has empty boundary, so there are only closed walk and cycle flow measurements, and the network is structured in such a way that the decorated type I and type II generating functions are a polynomial and a formal power series in $\mathbf{t}$, respectively.

In general, these generating functions will be a Laurent polynomial and a rational function in $\mathbf{t}$, respectively (Proposition 2.7 in [2]). In addition, on surfaces with nonempty boundary, there may be boundary path class measurements as well to consider. Therefore, a proper generalization of Theorem 2.5 must be able to account for boundary path class measurements (both individually and within flows). Fortunately there is a result of Talaska [3] that together with Theorem 2.5 allows us to approach this desired generalization. Here we paraphrase the relevant definition and theorem from her paper.

Definition 4.1 (Definition 2.1 in [3])**.** Let G be an edge-weighted directed graph, and fix an ordering $V(G) = \{v_1, \ldots, v_n\}$ of its vertices. The **weighted path matrix** of G is the matrix Λ whose entries ℓ_{ij} are the formal power series

$$\ell_{ij} = \sum_{p: v_i \rightsquigarrow v_j} \operatorname{wt}(p),$$

where the sum is over all directed paths from v_i to v_j.

Theorem 4.2 (Theorem 2.5 in [3])**.** *Suppose G is an edge-weighted directed graph with weighted path matrix Λ. Then the minor $\Delta_{X,Y}(\Lambda)$, with rows indexed by $X \subseteq V(G)$ and columns indexed by $Y \subseteq V(G)$, is given by*

$$\Delta_{X,Y}(\Lambda) = \frac{\sum_{F \in \mathcal{F}_{X,Y}(G)} \operatorname{sgn}(F) \operatorname{wt}(F)}{\sum_{F' \in \mathcal{F}_C(G)} \operatorname{sgn}(F') \operatorname{wt}(F')},$$

where $\mathcal{F}_{X,Y}(G)$ is the collection of flows whose components connect the vertices X to the vertices Y in G (i.e., path flows) and $\mathcal{F}_C(G)$ is the collection cycle flows on G.

We would like to apply this theorem to prove an analogue of Proposition 2.8 for general networks on surfaces; the following is a proposal of how to accomplish this.[5] First, let $\mathcal{N}$ be a network on a surface, and fix an ordering of its boundary vertices, $B = \{v_1, \ldots, v_n\}$. Define the **decorated weighted path matrix** for $\mathcal{N}$, written $\Lambda_H(\mathcal{N}, \mathbf{t})$, to be the matrix whose entries λ_{ij} are given by

$$\lambda_{ij} = \sum_{[p]: v_i \rightsquigarrow v_j} M^{[p]} \mathbf{t}^{\mathbf{h}[p]}.$$

Next we define two more generating functions in analogy to the decorated type II measurement generating function $\mathcal{M}^{\mathrm{II}}(\mathcal{N}, \mathbf{t})$ we defined in Section 2, except here we restrict our attention to specific collections of flows. First, we define the **decorated cycle flow measurement generating function** $\mathcal{M}^{\mathrm{II}}_C(\mathcal{N}, \mathbf{t})$ simply by only summing over cycle flows. Second, we define the **decorated** X, Y**-path flow measurement generating function** $\mathcal{M}^{\mathrm{II}}_{X \to Y}(\mathcal{N}, \mathbf{t})$ by only summing over path flows supported on the subsets $X \subseteq B^+, Y \subseteq B^-$.

Based on Theorem 4.2, we expect there to be a result of the following form: if $X \subseteq B^+$, $Y \subseteq B^-$, then the minor $\Delta_{X,Y}(\Lambda_H(\mathcal{N}, \mathbf{t}))$, with rows indexed by X and columns indexed by Y, is given by

$$\Delta_{X,Y}(\Lambda_H(\mathcal{N}, \mathbf{t})) = \frac{\mathcal{M}^{\mathrm{II}}_{X \to Y}(\mathcal{N}, \mathbf{t})}{\mathcal{M}^{\mathrm{II}}_C(\mathcal{N}, \mathbf{t})}.$$

By multiplying through by the denominator on the right-hand side and then using Proposition 2.8 (or an appropriate analogue), one could possibly extract algebraic relations among the coefficients of these generating functions that would allow one to imitate the proof of Theorem 2.1. In an ideal world, this approach would lead one to prove the following conjecture.

Conjecture 4.3. *If $\mathcal{N}$ is any network on any surface, $R^{\mathrm{I}}_H(\mathcal{N}) = R^{\mathrm{II}}_H(\mathcal{N})$ and $R^{\mathrm{I}}_U(\mathcal{N}) = R^{\mathrm{II}}_U(\mathcal{N})$.*

One final word of caution regarding this proposed approach. In all of the above manipulations of generating functions, one must make sure that the relations among the generating functions are formally meaningful. For instance, some of these relations may be nonsensical for generating functions that are only Laurent series in $\mathbf{t}$ (not power series, or polynomial). In considering this issue, we refer the reader to Conjecture 4.1 in [2], which may be useful.

4.2. **Generalizing Theorem 3.4.** In our proof of Theorem 3.4, we appealed to a result in [2] that allowed us to conclude that on an (n, m, k)-torus, the sign function is constant on flow classes. This allowed us to deduce that $M_H^{[F]} = M_H^{[F^c]}$, and hence that $R^{\mathrm{II}}_H(\mathcal{N})$ and $R^{\mathrm{II}}_U(\mathcal{N})$ are generated by the same set of measurements.

In general the sign function is *not* constant on flow classes, as the following example shows.

Example 4.4. In Figure 4, the two flows drawn in red connect the same sources and sinks, and they trivially define the same homology class; hence they are equivalent. However, the permutations they induce $B^+ \to B^-$ (from sources to sinks) differ by a single transposition, and so the two flows have different signs. Nevertheless, their complements (drawn in blue) also have different signs.

Perhaps the non-constancy of the sign function on flow classes on general networks seems to suggest that, in general, one must group flow measurements by a finer equivalence relation. However, the above example suggests that sign discrepancies may behave well under flow complementation. Namely, we are led to the following conjecture.

[5]In what follows, we define everything in terms of highway measurements, ommitting the subscript H to simplify notation. The corresponding definitions for underway measurements are completely analogous.

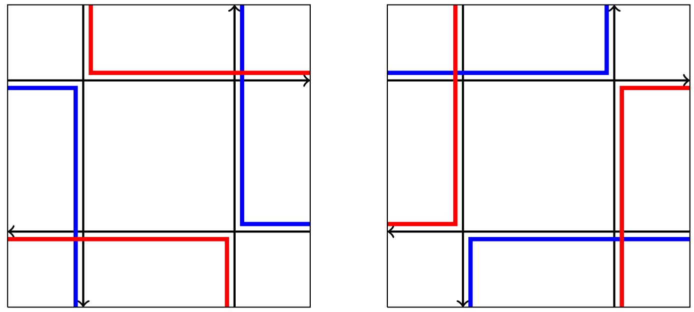


Figure 4. A network on a surface that is homeomorphic to a disk. Two equivalent flows are shown in red; their complement flows (also equivalent) are shown in blue.

Conjecture 4.5. *Let $\mathcal{N}$ be any network on any surface. If $F_1, F_2 \in \mathcal{F}(\mathcal{N})$ and $F_1 \sim F_2$, then*

$$\operatorname{sgn}(F_1)\operatorname{sgn}(F_2) = \operatorname{sgn}(F_1^{\mathsf{c}})\operatorname{sgn}(F_2^{\mathsf{c}}).$$

Let $\mathcal{N}$ be an arbitrary network on a surface. The truth of this conjecture would show that the relative parities (i.e., values of sgn) of flow measurements for the members of a given flow class is preserved under complementation. We would then deduce $M_H^{[F]} = \pm M_H^{[F^{\mathsf{c}}]}$ for any flow $F \in \mathcal{F}(\mathcal{N})$, and consequently the rings $R_H^{\mathrm{II}}(\mathcal{N})$ and $R_U^{\mathrm{II}}(\mathcal{N})$ would have the same set of generators (up to sign). In other words, we have observed that the previous conjecture implies the following one.

Conjecture 4.6. *Let $\mathcal{N}$ be any network on any surface. Then $R_H^{\mathrm{II}}(\mathcal{N}) = R_U^{\mathrm{II}}(\mathcal{N})$.*

Proving this conjecture as well as Conjecture 4.3 would in turn resolve the mirror symmetry conjecture for all networks on surfaces.

References

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