Factorizations of k-Nonnegative Matrices

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Background

Introduction

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- The space of invertible totally nonnegative matrices form a semigroup
- Fomin & Zelevinsky nicely characterize and parametrize the semigroup via factorization-based cells

Loewner-Whitney Theorem

The subsemigroup of invertible TNN upper unitriangular matrices has generating set $\{e_i(a) \mid i \in [n-1], a > 0\}$ (a is the *parameter*):

$$e_i(a) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & a & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

Relations

$$
\begin{aligned}
e_i(a)e_i(b) &= e_i(\alpha) \\
e_i(a)e_{i+1}(b)e_i(c) &= e_{i+1}(\alpha)e_i(\beta)e_{i+1}(\gamma) \\
e_i(a)e_j(b) &= e_j(\alpha)e_i(\beta) \qquad |i-j| > 1
\end{aligned}
$$

The conversion expression for all parameters is *subtraction-free*, and for the latter two, *bijective*.

Defining Factorizations

Define the free word monoid $\mathcal{A} = \langle e_i \mid i \in [n-1] \rangle$ and define an equivalence relation generated by

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Define a *length function* $\ell : \mathcal{A} \to \mathbb{N}$ to be the number of letters in a word. Define the *parameter map* for a word $w \in \mathcal{A}$ by

$$x_w : \mathbb{R}_{>0}^{\ell(w)} \to GL_n(\mathbb{R}) \qquad x_w(a_1, \ldots, a_{\ell(w)}) = w_1(a_1) \cdots w_{\ell(w)}(a_{\ell(w)})$$

Thus, the image of the parameter map is the set of matrices with w as a factorization.

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By noticing the identification $e_i \mapsto (i, i+1) \in S_n$ which generate S_n as a Coxeter group, we see:

- the cells are naturally indexed by elements in S_n
- the cells form a CW-complex
- the corresponding closure poset is isomorphic to the Bruhat poset on S_n

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- Invertible k-nonnegative matrices form a semigroup
- Our new work attempts to generalize TNN results to the kNN case
- We succeed in two cases: $(n-1)$NN matrices, and $(n-2)$NN unitriangular matrices

Results

Generators

Theorem

The semigroup of $(n-2)NN$ upper unitriangular matrices is generated by the e_i's and the T-generators.

The T-generators have the following form.

$$T(\vec{a},\vec{b}) = \begin{bmatrix} 1 & a_1 & a_1 b_1 & & & & & \\ & 1 & a_2+b_1 & a_2 b_2 & & & & \\ & & 1 & \ddots & \ddots & & & \\ & & & \ddots & a_{n-3}+b_{n-4} & a_{n-3}b_{n-3} & & \\ & & & & 1 & b_{n-3} & b_{n-2}Y \\ & & & & & 1 & b_{n-2}X \\ & & & & & & 1 \end{bmatrix}$$

$$Y = b_1 \cdots b_{n-3} \qquad X = \left| T_{[2,n-3],[3,n-2]} \right|$$

Relations

Adding T leads to additional relations. The following is a complete list (indices are mod $n-1$):

- $e_i(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_{i+2}(x')$
- $e_{n-1}e_{n-2}T =$
  $e_{n-2}e_{n-1}T \sqcup e_{n-2}\cdots e_1 e_{n-1}\cdots e_2 \sqcup e_{n-2}\cdots e_1 e_{n-1}\cdots e_1$

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These relations are *bijective* and *subtraction-free* as desired.

We can extend the parameter map x_w and thus $U(w)$ to add T.

Reduced Words

Consider the alphabet $\mathcal{B} = \langle e_1, e_2, \ldots, e_{n-1}, T\rangle$ modulo all relations.

Theorem

Let $w_{0,[n-2]} = (n-2, n-3, \ldots, 1, n-1, n)$ in one-line notation. Then all words with at most one T are equal to one of the following distinct reduced words:

$$\begin{cases} v\lambda & v \leq w_{0,[n-2]}, \\ & \lambda \in \{T, e_{n-1}T, e_{n-2}T, e_{n-2}e_{n-1}T\} \\ w & w \in S_n \end{cases}$$

Cell Topology

Theorem (Disjointness)

For reduced words v and w, if $v \neq w$ then $U(v) \cap U(w) = \emptyset$.

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Theorem (Well-behaved Closure Order)

The closure of a cell $\overline{U(w)}$ is the disjoint union of all cells in the interval between $\emptyset$ and $U(w)$ subject to the subword order on $\mathcal{B}$.

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Corollary (CW-complex)

The set of $U(w)$ form a CW-complex, with closure relations described above.

Subword Order

We can still describe the cell closure poset with a subword order. To do this, we extend the Bruhat order on S_n by defining the subwords of T.

- $m < \lambda \in \{T, e_{n-1}T, e_{n-2}T,\ e_{n-2}e_{n-1}T\}$ precisely when $m \leq \alpha = e_{n-2} \cdots e_1 e_{n-1} \cdots e_1$ and satisfies the following:
- $m(1) \neq n$; if λ has no e_{n-1}, then $m(2) \neq n$ is relaxed; if λ has no e_{n-2}, then $m(1) \neq n-1$.

This description still defines a valid subword order.

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See our preprint at arXiv:1710.10867 for more information.

Cell Topology (cont.)

Theorem

The poset on $\{U(w)\}$ given by the subword order on reduced words is graded.

What other properties does the closure poset attain? More knowledge would lead to understanding the shape of the space.

We know that the space is not a sphere, since the poset is not Eulerian.

General TNN Matrices

General TNN matrices are generated by $e_i(a)$'s, $e_i(a)^T$'s, and diagonal matrices. They are parametrized via *double Bruhat cells*.

The poset of closure relations between double Bruhat cells is isomorphic to Bruhat order on the Coxeter group $S_n \times S_n$.

Generators of $(n-1)$-Nonnegative $n \times n$ Matrices

Theorem

The semigroup of $(n-1)$-nonnegative matrices is generated by the e_i's, e_i^T's, diagonal matrices, and the K-generators.

The K-generators have the following form.

$$K(\vec{a}, \vec{b}) = \begin{bmatrix} a_1 & a_1 b_1 & & & & \\ 1 & a_2 + b_1 & a_2 b_2 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & a_{n-2} + b_{n-3} & a_{n-2} b_{n-2} & \\ & & & 1 & b_{n-2} & b_{n-1} Y \\ & & & & 1 & b_{n-1} X \end{bmatrix}$$

$$Y = b_1 \cdots b_{n-2}$$

$$X = \left| K_{[2,n-2],[2,n-2]} \right| = \sum_{k=1}^{n-2} \left(\prod_{\ell=2}^{k} b_{\ell-1} \prod_{\ell=k+1}^{n-2} a_\ell \right)$$

Cells of $(n-1)$-Nonnegative Matrices

- K behaves well with other generators, giving similar relations as before
- A similar reduced word scheme can be made using the alphabet

$$\mathcal{S} = \{1, \ldots, n-1, \textcircled{1}, \ldots, \textcircled{n}, \overline{1}, \ldots, \overline{n-1}, K\}$$

- This gives cells homeomorphic to open balls which partition the space and whose closure relations are equivalent to taking subwords
- The space does not form a CW-complex, since it consists of two connected components: matrices with positive determinant and matrices with negative determinant