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REU 2018 Day 2 Vic Reiner

Binary matroids & sandpile groups

1. Counting trees
2. Sandpile group
3. Cayley graphs for $\mathbb{F}_2^r$
4. REU Problem 2
5. Ring theory

1. Counting trees (see Loehr §3.3, Stanley Chap. 9)

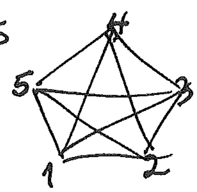
$G = (V, E)$ an undirected graph with no self-loops 

V = vertices, E = edges

but parallel edges OK 

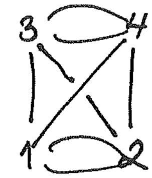
e.g. K_n = complete graph on n vertices

$n=5$



K_5

e.g.

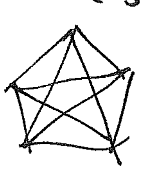
$G=$ 

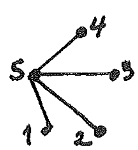
A spanning tree in G is a subset $T \subseteq E$ that connects all of V and contains no cycles

$\tau(G) \overset{\text{DEF'N}}{:=}$ # of spanning trees in G (=0 if G is disconnected)

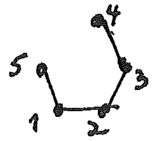
e.g. THEOREM (Cayley 1889, Borchardt 1860) $\tau(K_n) = n^{n-2}$

$\tau(K_5) = 5^3 = 125$

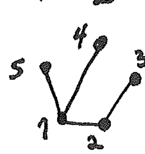




5 such

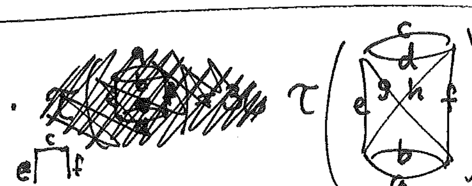


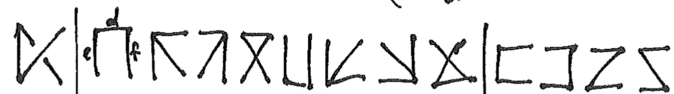
$\frac{5!}{2} = 60$ such



$5 \cdot 4 \cdot 3 = 60$ such

125 total

e.g.  $= 36$



4 | 2 2 2 2 2 2 2 2 | 4 4 4 4

16 | 16

36 total

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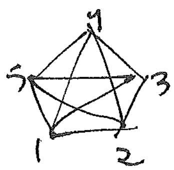
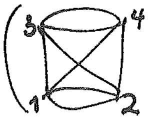
$\tau(G)$ is easier to compute in 2 ways via the

<u>graph Laplacian</u> $L(G) \in \mathbb{Z}^{n\times n}$ where $n = |V|$

$$L(G)_{i,j} = \begin{cases} \deg_G(i) & \text{if } i=j \\ -(\#\text{edges } i - j) & \text{if } i\neq j \end{cases}$$

e.g. $L(K_5) = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \end{array}$



$L\left(\right.$ 

$\left.\right) = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 4 & -2 & -1 & -1 \\ -2 & 4 & -1 & -1 \\ -1 & -1 & 4 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix} \end{array}$

Note $L(G)$ is <u>singular</u> always because $\begin{bmatrix}1\\1\\ \vdots \\1\end{bmatrix}$ is in its nullspace

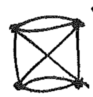
$\overline{L(G)}^{i,i} :\overset{\text{DEF'N}}{=} L(G) - \{ i^{\text{th}} \text{ row}, i^{\text{th}} \text{ column} \}$

THEOREM:

(a) (Kirchhoff 1847 Matrix Tree Thm) $\tau(G) = \det\left(\overline{L(G)}^{i,i}\right)$

(b) (eigenvalue version; see Stanley COR 9.10) If $L(G)$ has eigenvalues $(0=)\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

then $\tau(G) = \dfrac{\lambda_2 \lambda_3 \cdots \lambda_n}{n}$

e.g. $\tau\left(\right.$  $\left.\right)$ $= \det \overline{L(G)}^{4,4} = \det \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 4 & -2 & -1 \\ -2 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix} \end{array} = 36$

G

Can be computed fast, in $< cn^3$ steps via Gaussian elimination

or alternatively, $L(G)$ has eigenvalues $\begin{matrix} \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \\ \| \quad \| \quad \| \quad \| \\ 0 \quad 4 \quad 6 \quad 6 \end{matrix} \Rightarrow \tau(G) = \dfrac{4\cdot 6\cdot 6}{4} = 36$

Try this in SAGEMATH CELL (or in COCALC):

```
L=matrix([[4,-2,-1,-1],
          [-2,4,-1,-1],
          [-1,-1,4,-2],
          [-1,-1,-2,4]])
L.eigenvalues()
```

REU EXERCISE 4: (a) Show $L(K_n) = nI_n - J_n$ where $J_n = n\times n$ all ones matrix $\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$

(b) Explain why J_n has eigenvalues $(n, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}})$

(c) Prove $\tau(K_n) = n^{n-2}$.

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2. Sandpile group

For G connected, $\det \overline{L(G)}^{i,i} = \tau(G) \neq 0$ shows $\mathbb{R}^n \xrightarrow{L(G)} \mathbb{R}^n$ has rank $n-1$, so has

$$\ker L(G) \cong \mathbb{R}^1$$
$$\operatorname{im} L(G) \cong \mathbb{R}^{n-1}$$
$$\operatorname{coker} L(G) := \mathbb{R}^n / \operatorname{im} L(G) \cong \mathbb{R}^1.$$

But $L(G) \in \mathbb{Z}^{n\times n}$, so what about as a map $\mathbb{Z}^n \xrightarrow{L(G)} \mathbb{Z}^n$?

Not hard to see $\ker L(G) \cong \mathbb{Z}^1$
$\operatorname{im} L(G) \cong \mathbb{Z}^{n-1}$

but $\operatorname{coker} L(G) := \mathbb{Z}^n / \operatorname{im} L(G)$ is interesting:

$$\cong \mathbb{Z} \oplus \underbrace{K(G)}_{\text{sandpile group of } G\text{, a finite abelian group}}$$

i.e. $K(G) \cong \bigoplus_{\substack{i=1 \\ d_i \text{ dividing } d_{i+1}}}^{n-1} \mathbb{Z}/d_i\mathbb{Z} \cong \bigoplus_{\text{primes } p} \bigoplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m_{p,e}}$

$\boxed{\text{Equivalently, } K(G) \cong \operatorname{coker}(\overline{L(G)}^{i,i})}$
$\boxed{\Rightarrow |K(G)| = \tau(G)}$

One can compute $K(G)$ by making change of $(\mathbb{Z}\text{-})$bases in $\mathbb{Z}^n \xrightarrow{L(G)} \mathbb{Z}^n$ that put $L(G)$ into Smith normal form: $P\,L(G)\,Q = \begin{bmatrix} d_1 & & & & \\ & d_2 & & 0 & \\ & & \ddots & & \\ & 0 & & d_{n-1} & \\ & & & & 0 \end{bmatrix}$ with $d_i \mid d_{i+1}$

for $P, Q \in GL_n(\mathbb{Z})$

row ops over $\mathbb{Z}$ — column ops over $\mathbb{Z}$

e.g.

$L($$) = \begin{bmatrix} 4 & -2 & -1 & -1 \\ -2 & 4 & -1 & -1 \\ -1 & -1 & 4 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix} \overset{\text{row, col ops}}{\rightsquigarrow} \begin{bmatrix} 4 & -2 & -1 & 0 \\ -2 & 4 & -1 & 0 \\ -1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & -6 & 15 & 0 \\ 0 & 6 & -9 & 0 \\ -1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & -9 & 0 \\ 0 & -6 & 15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightsquigarrow \begin{bmatrix} 1 & & & \\ & 6 & -9 & \\ & 0 & 6 & \\ & & & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & & & \\ & 6 & -3 & \\ & 0 & 6 & \\ & & & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & & & \\ & 0 & -3 & \\ & 12 & 6 & \\ & & & 0 \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 1 & & & \\ & 3 & & \\ & & 12 & \\ & & & 0 \end{bmatrix}}_{= \text{Smith form}}$

$\Leftarrow$ $\operatorname{coker} L(G) = \mathbb{Z} \oplus K(G) = \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/0\mathbb{Z}$
$= \mathbb{Z} \oplus \underbrace{\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}}$

$K(G) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$

$\boxed{\text{Try in SAGE (COCALC): } \texttt{L.smith\_form()}}$

$(\Longrightarrow \operatorname{rank}_{\mathbb{F}_2} L(G) = 2,\ \operatorname{rank}_{\mathbb{F}_3} L(G) = 1.$ Why?$)$

3. Cayley graphs for $\mathbb{F}_2^r$

Sometimes knowing eigenvalues of $L(G)$ and hence $\tau(G)$ gives us a guess for the structure of the abelian group $K(G)$ (having $|K(G)| = \tau(G)$).

e.g. K_n has $\tau(K_n) = n^{n-2}$

and $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$ is not too hard to show by various means

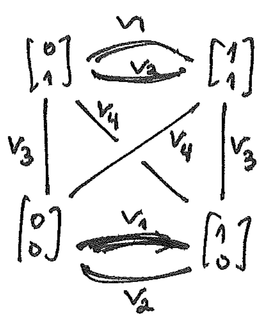
DEF'N: Given a group Γ and generating set $M = \{m_1, \ldots, m_n\}$ which are involutions ($m_i^2 = e$), define the Cayley graph $G(\Gamma, M)$ to have vertex set $V = \Gamma$

edge set $E = \{ \{g, gm_i\} \; g \in G, m_i \in M \}$

$g \xrightarrow{v_i} gm_i$

e.g. $\Gamma = \mathbb{F}_2^2$ and $M =$ columns of this matrix

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\rightsquigarrow$ Cayley graph $G(\Gamma, M) =$ 

$= G$ from before

It turns out there is an eigenbasis for all of these $G(\mathbb{F}_2^r, M)$ Cayley graphs at once .... (see Stanley Chap. 2)

DEF'N: Given $u, v \in \mathbb{F}_2^r$, let $u \cdot v = u_1v_1 + \cdots + u_nv_n \in \mathbb{F}_2$, and $(-1)^{u \cdot v} = \begin{cases} +1 & \text{if } u \cdot v = 0 \\ -1 & \text{if } u \cdot v = 1 \end{cases}$

Define for $u \in \mathbb{F}_2^r$ the vector $f_u \in \mathbb{F}_2^{2^r}$ with basis $\{e_x\}_{x \in \mathbb{F}_2^r}$ by $(f_u)_x = (-1)^{u \cdot x}$, or equivalently, $f_u = \sum_{x \in \mathbb{F}_2^r} (-1)^{u \cdot x} e_x$

e.g.

$f_{\begin{bmatrix}0\\0\end{bmatrix}}$	$f_{\begin{bmatrix}1\\0\end{bmatrix}}$	$f_{\begin{bmatrix}0\\1\end{bmatrix}}$	$f_{\begin{bmatrix}1\\1\end{bmatrix}}$
$+1\begin{bmatrix}0\\1\end{bmatrix} \cdots \begin{bmatrix}1\\1\end{bmatrix}+1$	$+1\begin{bmatrix}0\\1\end{bmatrix} \cdots \begin{bmatrix}1\\1\end{bmatrix}-1$	$-1\begin{bmatrix}0\\1\end{bmatrix} \cdots \begin{bmatrix}1\\1\end{bmatrix}-1$	$-1\begin{bmatrix}0\\1\end{bmatrix} \cdots \begin{bmatrix}1\\1\end{bmatrix}+1$
$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$
$+1\begin{bmatrix}0\\0\end{bmatrix} \cdots \begin{bmatrix}1\\0\end{bmatrix}+1$	$+1\begin{bmatrix}0\\0\end{bmatrix} \cdots \begin{bmatrix}1\\0\end{bmatrix}-1$	$+1\begin{bmatrix}0\\0\end{bmatrix} \cdots \begin{bmatrix}1\\0\end{bmatrix}+1$	$+1\begin{bmatrix}0\\0\end{bmatrix} \cdots \begin{bmatrix}1\\0\end{bmatrix}-1$

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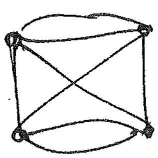
REU Exercise 5

(a) Prove that $\{f_u\}_{u \in \mathbb{F}_2^r}$ give an orthogonal basis for $\mathbb{R}^{2^r}$, and that the <u>standard</u> basis $\{e_u\}_{u\in\mathbb{F}_2^r}$ for $\mathbb{R}^{2^r}$ satisfies $e_u = \frac{1}{2^n} \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} f_v$

(b) Show that for <u>any</u> set $M = \begin{array}{c} \begin{matrix} v_1 & v_2 & \cdots & v_n \end{matrix} \\ \begin{bmatrix} \vdots & \vdots & & \vdots \end{bmatrix} \end{array} = \{v_1, v_2, \ldots, v_n\}$ of generators of $\mathbb{F}_2^r$, the Cayley graph $G = G(\mathbb{F}_2^r, M)$ and its graph Laplacian $L(G)$ have every f_u as <u>eigenvector</u>, with eigenvalue $\lambda_{u,M} = n - \sum_{i=1}^{n} (-1)^{u \cdot v_i}$ (meaning $L(G) f_u = \lambda_{u,M} \cdot f_u$)

e.g. $M = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$

$n = 4$

$G(\mathbb{F}_2^2, M) =$ 

$u =$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
eigenvalue $\lambda_{u,M} =$	$4-1-1-1-1 = 0$	$4+1+1-1+1 = 6$	$4-1-1+1+1 = 4$	$4+1+1+1-1 = 6$

i.e. eigenvalues $(0, 4, 6, 6)$ as before

(c) Show that if we define the ring $R := \mathbb{Z}[\frac{1}{2}] := \{ \frac{a}{2^\ell} : a \in \mathbb{Z},\ \ell = 0, 1, 2, \ldots \}$ (= "$\mathbb{Z}$ localized away from the prime 2")

so that

$$\mathbb{Z} \subset R \subset \mathbb{Q},$$

then $\exists$ an R-basis for R^{2^r} in which $R^{2^r} \xrightarrow{L(G)} R^{2^r}$ acts <u>diagonally</u>, with eigenvalues $\{\lambda_{u,M}\}_{u \in \mathbb{F}_2^r}$

(d) Explain why this shows $G = G(\Gamma, M)$ always has

$$K(G) \cong \bigoplus_{\text{primes } p} \underbrace{\bigoplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m_{p^e}}}_{=: \mathrm{Syl}_p K(G)}$$

satisfying $\mathrm{Syl}_p K(G) \cong \mathrm{Syl}_p\left(\bigoplus_{u \in \mathbb{F}_2^r - \{0\}} \mathbb{Z}/\lambda_{u,M}\mathbb{Z} \right)$ for <u>odd</u> primes p.

e.g. $K($  $) \cong (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z}$ has $\mathrm{Syl}_3 K(G) \cong (\mathbb{Z}/3\mathbb{Z})^2 \cong \mathrm{Syl}_3(\underbrace{\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}}_{(\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2})$

4. REU PROBLEM 2

Describe the structure of the rest of the sandpile groups $K(G)$ where $G = \Gamma(\mathbb{F}_2^r, M)$ with M = columns of $r\left\{\begin{bmatrix} 1 & 1 & & 1 \\ v_1 & v_2 & \cdots & v_n \\ 1 & 1 & & 1 \end{bmatrix}\right. \in \mathbb{F}_2^{r\times n}$,

namely $\underline{\underline{Syl_2 K(G)}}$, in terms of the matrix M.

E.g.
- How many generators does it need?
- Is there a good bound on e in $Syl_2 K(G) = \bigoplus_{e\geq 1} (\mathbb{Z}/2^e\mathbb{Z})^{m_{2^e}}$?
- How does it depend on the matroid of M, that is, data about M that doesn't vary if we multiply $M \mapsto PM$ for some $P \in GL_r(\mathbb{F}_2)$?

Another good feature is that ring theory can be applied here...

e.g. for $G = G(\mathbb{F}_2^2, M)$ as before, we can model $\mathbb{Z}^{2^2}$ as $\mathbb{Z}[x_1,x_2]/(x_1^2-1, x_2^2-1)$

$$M = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline x_1 & 1 & 1 & 0 & 1 \\ x_2 & 0 & 0 & 1 & 1 \end{array}$$

with $\mathbb{Z}$-basis

$$\begin{array}{cccc} \{1, & x_1, & x_2, & x_1x_2\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \begin{bmatrix}0\\0\end{bmatrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}0\\1\end{bmatrix} & \begin{bmatrix}1\\1\end{bmatrix} \end{array}$$

and then model $\mathbb{Z}^{2^2} \xrightarrow{L(G)} \mathbb{Z}^{2^2}$ as multiplication by $4 - (\underset{v_1,v_2}{2x_1} + \underset{v_3}{x_2} + \underset{v_4}{x_1x_2})$, so that

$$\mathbb{Z} \oplus K(G) \cong \operatorname{coker} L(G) \cong \mathbb{Z}[x_1,x_2]/\underbrace{(x_1^2-1, x_2^2-1, 4-(2x_1+x_2+x_1x_2))}_{I}$$

computed a Gröbner basis over $\mathbb{Z}$ for the ideal I in SAGE

$$\overset{\text{as } \mathbb{Z}\text{-modules}}{\cong} \mathbb{Z}[x_1,x_2]/(\underline{x_1^2}-1, \underline{x_1x_2}+2x_1+x_2-4, \underline{3x_1}+6x_2-9, \underline{12x_2}-12, \underline{x_2^2}-1,)$$

$$\cong \underset{\text{spanned by } 1}{\mathbb{Z}} \oplus \underset{x_1}{\mathbb{Z}/3\mathbb{Z}} \oplus \underset{x_2}{\mathbb{Z}/12\mathbb{Z}}$$

Try in SAGE:

```
R.<x1,x2>=ZZ[]
I=R.ideal([x1^2-1,x2^2-1,4-(2*x1+x2+x1*x2)])
I.groebner_basis()
```

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REU Exercise 6:

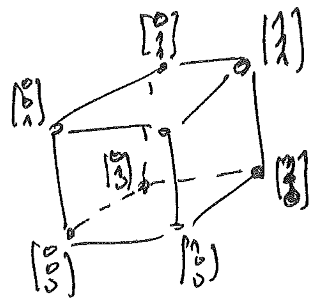
Show that $G = G(\mathbb{F}_2^r, M)$ for $M = r\left\{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}\right. \in \mathbb{F}_2^{r \times n}$

has $\operatorname{coker} L(G) \cong \mathbb{Z}[x_1, x_2, \ldots, x_r] / \left(x_1^2 - 1, \ldots, x_r^2 - 1,\ n - \sum_{i=1}^{n} x_1^{(v_i)_1} x_2^{(v_i)_2} \cdots x_r^{(v_i)_r}\right)$
$(= \mathbb{Z} \oplus K(G))$

For the special case where $M = \begin{array}{c} \overbrace{}^{r} \\ r\left\{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \ 1 \end{bmatrix}\right. \end{array}$

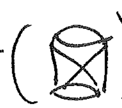
then $G = (\mathbb{F}_2^r, M) = r$-cube graph Q_r

Q_1: $[0] — [1]$ | Q_2:
$$\begin{tikzcd} \begin{bmatrix}0\\1\end{bmatrix} \arrow[r, dash] \arrow[d, dash] & \begin{bmatrix}1\\1\end{bmatrix} \arrow[d, dash] \\ \begin{bmatrix}0\\0\end{bmatrix} \arrow[r, dash] & \begin{bmatrix}1\\0\end{bmatrix} \end{tikzcd}$$
| Q_3



and a lot of work has been done, but $\mathrm{Syl}_2\, K(Q_r)$ is still not known!

- see paper of H. Bai for partial results and data
- see REU report of Anzis–Prasad for ring approach
- see paper of Chandler–Sin–Xiang for $\operatorname{coker} A(G)$ instead of $\operatorname{coker} L(G)$, where $A(G)$ = adjacency matrix

$A($  $) = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

which turns out to be ~~totally~~ predictable from the eigenvalues!