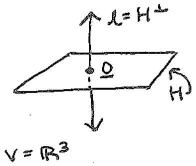
Weight polytopes (= Wythoff's Construction): 6/19/18

1. Reflection & Coxeter groups
2. Weight polytopes
3. simple polytopes & f-vectors
4. REU problem #7

1. Reflection Groups

Def: A reflection t acting on $V = \mathbb{R}^n$ is an element $t \in GL_n(\mathbb{R})$ that fixes a hyperplane H and negates the line $H^\perp$ (perpendicular to H).

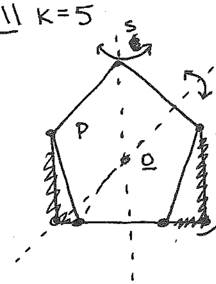
i.e., co-dim = 1
dim = n-1
linear subspace
(through $\underline{0}$)



H is called the reflection hyperplane for t.

Def: A finite reflection group W is a finite subgroup $W \subset GL_n(\mathbb{R})$ generated by reflections.

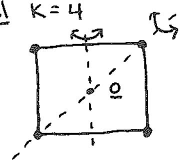
EX $k=5$



$t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$W = I_2(k)$ for $k \geq 3$
= "dihedral group of order $2k$"
:= linear symmetries of a regular k-sided polygon P
$\left(:= \{ g \in GL_n(\mathbb{R}) : g(P) = P \} \right)$

EX $k=4$



REU Exercise 17:

(a) Prove $I_2(k) = \underbrace{\{e, r, r^2, \ldots, r^{k-1}\}}_{\text{rotations}} \sqcup \underbrace{\{s, sr, sr^2, \ldots, sr^{k-1}\}}_{\text{reflections}}$
$\quad\; \cap$
$GL_2(\mathbb{R})$

(b) Prove the abstract presentation $I_2(k) \cong \langle s, r : s^2 = e = r^k,\ srs = r^{-1} \rangle$

(c) Prove the Coxeter presentation $I_2(k) \cong \langle s, t : s^2 = t^2 = e,\ (st)^k = e \rangle$

Hint: create well-defined maps between presentations, satisfying the necessary relations. i.e., for (b), where should you map s, r?
(part (a) gives you surjectivity for your argument)

Def: A Coxeter presentation for a group W is of the form

$$W \cong \langle \overbrace{\{s_1, \ldots, s_n\}}^{S} \mid s_i^2 = e,\ (s_i s_j)^{m_{ij}} = e \rangle$$

for some $m_{ij} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$

These presentations can be encapsulated in a (Coxeter) group for (W, S):

- vertices = S
- edges = $(s_i) \overset{m_{ij}}{—} (s_j)$ with edges labeled $m_{ij} = 2$ ommitted, edges labeled $m_{ij} = 3$ unlabeled

EX $I_2(k)$ has Coxeter diagram $(s) \overset{k}{—} (t)$

e.g.,

$\longleftrightarrow I_2(5) \longleftrightarrow (s) \overset{5}{—} (t)$

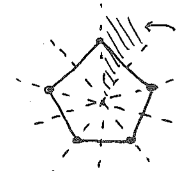
Which finite groups have a Coxeter presentation?

Theorem: (Coxeter) Finite reflection groups W ($\subset GL_n(\mathbb{R})$) always have one, specifically by letting

$$S = \left\{ \begin{array}{l} \text{reflections } s_1, \ldots, s_n \text{ through hyperplanes bounding} \\ \text{a particular chamber cut out by all reflection hyperplanes} \end{array} \right\}$$

chamber ← connected component of $V - \bigcup H$ (reflection hyperplanes)

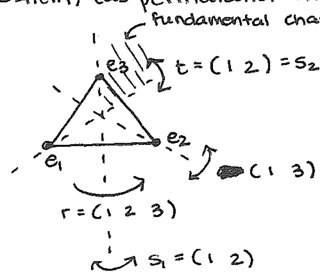
EX

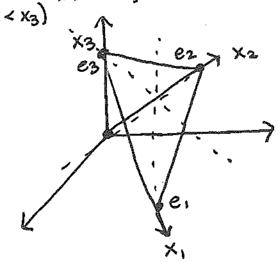


Remark: Conversely, if W has a Coxeter presentation and W is finite, then W is a reflection group ($\subset GL_n(\mathbb{R})$)

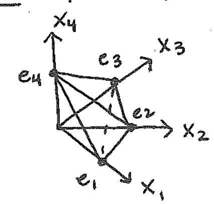
EX S_n = symmetric group on n letters really is a finite reflection group acting on $\mathbb{R}^n$, permuting coordinates.
$\cap$
$GL_n(\mathbb{R})$ (as permutation matrices)

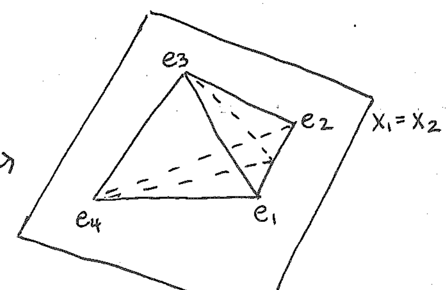
in $\mathbb{R}^3$,





EX $S_4 \subset GL_4(\mathbb{R})$

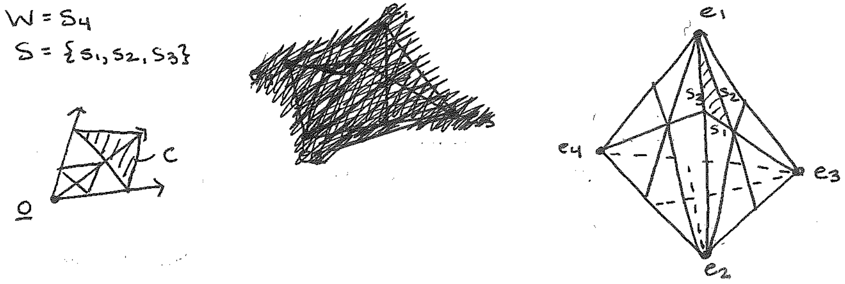




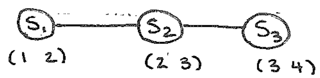
S_4 acts by reflection symmetries.

The hyperplanes / chambers of S_4 cuts out the <u>Coxeter complex</u> on the boundary of the tetrahedron.

<u>EX</u> $W = S_4$
$S = \{s_1, s_2, s_3\}$



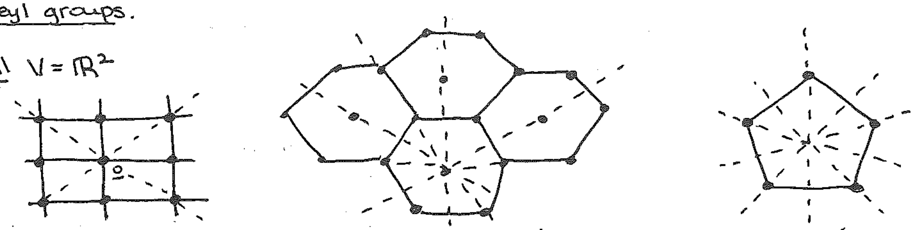
Corresponds to the Coxeter diagram



More generally, for $W = S_n$, $S = \{s_1, \ldots, s_{n-1}\}$
$(s_i s_j)^2 = e$ if $|i-j| \geq 2$
$(s_i s_{i+1})^3 = e$
which has Coxeter diagram of type A_{n-1}.

Some finite reflection groups W acting on $V = \mathbb{R}^n$ stabilize a full rank lattice $\Lambda \cong \mathbb{Z}^n$ inside $\mathbb{R}^n$, and are called <u>crystallographic</u> reflection groups, or <u>Weyl groups</u>.

<u>EX</u> $V = \mathbb{R}^2$



$I_2(4) = B_2/C_2$ $\qquad$ $S_3 = I_2(3) \subset I_2(6) = G_2$ $\qquad$ $I_2(5)$ (not crystallographic)

<u>REU Exercise #18</u>:
Show a Coxeter system (W, S) with W crystallographic must have all $m_{ij} \in \{2, 3, 4, 6\}$

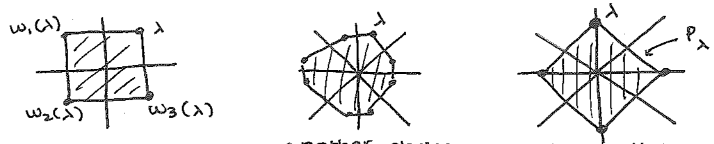
<u>Remark</u>: If (W,S) has W finite and all $m_{ij} \in \{2,3,4,6\}$, then W is a Weyl group.

<u>Remark</u>: Weyl groups have associated (linear) algebraic groups, like $G = GL_n(F)$ for $W = S_n$, with Borel subgroups B (= upper Δs for GL_n) and Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$.
W "controls" the representation theory and structure of G.

2. <u>Weight polytopes</u> (= Wythoff's construction)

<u>Def</u>: Given W a finite reflection group acting on $V = \mathbb{R}^n$, and pick a $\lambda \in V$, then
P_λ = weight polytope for λ
:= <u>convex hull</u> of the <u>W-orbit</u> of λ
smallest convex containing the $w(\lambda)$ $\qquad$ $\{w(\lambda)\}_{w \in W}$

<u>EX</u> $W = I_2(4) = B_2/C_2$

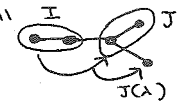


Every λ has a unique W-orbit representative inside chamber C, whose walls give $S = \{s_1, \ldots, s_n\}$.

If we let $J(\lambda) := \{s \in S : s(\lambda) = \lambda\}$ then $J(\lambda)$ and (W,S) control the facial structure of P_λ.

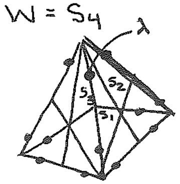
<u>Theorem</u>: (Renner 2009, Cor 1.3, for Weyl groups W; Maxwell more generally)

- P_λ has exactly one W-orbit of faces for each $I \subseteq S$ s.t. no connected component of I lies entirely in $J(\lambda)$.
  Call the set of such I's $\mathcal{S}(I)$.

(EX )

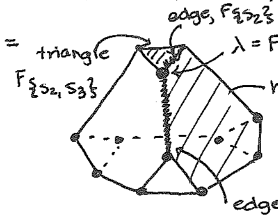
- This W-orbit is represented by a face F_I whose relative interior intersects $\bar{C}$ and has the <u>parabolic subgroup</u> $W_I = \langle s \rangle_{s \in I}$ stabilizing F_I, but acting non-trivially on F_I.
- The W-stabilizer of F_I is W_{I^*}, where $I^* = I \cup \{s \in J(\lambda) : st = ts \ \forall t \in I\}$

<u>EX</u> $W = S_4$



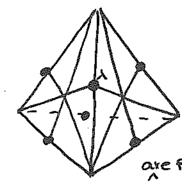
$J(\lambda) = \{s_3\}$

$\Rightarrow \mathcal{S}(\lambda) = \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}\}$

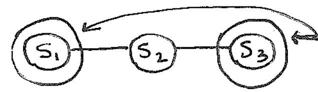
$P_\lambda =$


F_S is the <u>entire polytope</u>, P_λ

<u>EX</u> $W = S_4$

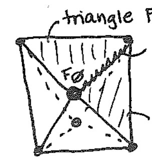


(marked points are for a new choice of λ)



$J(\lambda) = \{s_1, s_3\}$

$\Rightarrow \mathcal{S}(\lambda) = \{\emptyset, \{s_2\}, \{s_1, s_2\}, \{s_2, s_3\}, \overset{S}{\overset{\shortparallel}{\{s_1, s_2, s_3\}}}\}$

$P_\lambda =$


and $F_S = P_\lambda$.

3. <u>Simple polytopes & f-vectors</u>

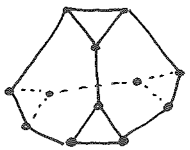
<u>Def</u>: For a convex polytope P of dimension n, its <u>f-vector</u> is $f(P) := (f_0, f_1, \ldots, \overset{1}{\overset{\shortparallel}{f_n}})$
where $f_i :=$ # of i-dim faces of P

<u>EX</u> Our first choice of λ for $W = S_4$ gave us P_λ with
$f(P_\lambda) = (\underset{\text{vertices}}{12}, \overset{\text{edges}}{18}, \underset{\text{facets}}{8}, 1)$

Since the W-orbit of F_I in P_λ (for $I \in \mathcal{S}(\lambda)$) looks like cosets W/W_{I^*} where W_{I^*} = W-stabilizer of F_I and has size $|W/W_{I^*}| = |W|/|W_{I^*}| = [W : W_{I^*}]$, $\dim(F_I) = |I|$.

Corollary: $f_i(P_\lambda) = \sum_{\substack{I \in \mathcal{S}_\lambda, \\ |I| = i}} \frac{|W|}{|W_{I^*}|}$

EX!!

$P_\lambda =$ 

(for $W = S_4$)

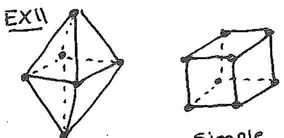
(each vertex has exactly n incident edges)

I	I^*	$\|W\|/\|W_{I^*}\|$	
$\emptyset$	S_3	$4!/2 = 12 = f_0$	
S_1	S_1, S_3	$4!/2\cdot 2 = 6$	$\}\ 18 = f_1$
S_2	S_2	$4!/2 = 12$ ✓	
S_1, S_2	S_1, S_2	$4!/3! = 4$	$\}\ 8 = f_2$
S_2, S_3	S_2, S_3	$4!/3! = 4$	
S_1, S_2, S_3	S	$4!/4! = 1 = f_3$	

When P is a simple n-dimensional polytope, there's a better way to encode the f-vector as the h-vector $h(P) = (h_0, h_1, \dots, h_n)$ s.t.

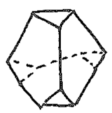
$h(P_\lambda, t) :=$

$$h_0 + h_1 t + h_2 t^2 + \cdots + h_n t^n = f_0 + f_1(t-1) + f_2(t-1)^2 + \cdots + f_n(t-1)^n$$

EX!! 

not simple   simple

For instance,

 has $f = (12, 18, 8, 1)$

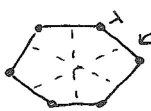
$\rightsquigarrow h = (1, 5, 5, 1)$

since $12 + 18(t-1) + 8(t-1)^2 + t^3 = 1 + 5t + 5t^2 + t^3$

For P simple,

- $h(P)$ is always symmetric ($h_i = h_{n-i}$)
- $h_i \geq 0$
- h_i have various algebraic & topological interpretations

EX!! $W = S_n$ and λ generic



$\leftarrow P_\lambda$ is called a permutahedron

$h(P_\lambda, t) = E_n(t)$ is called the Eulerian polynomial

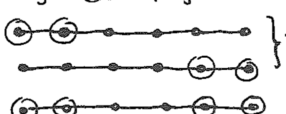
$= \sum_{w \in S_n} t^{\#\{i : w(i) > w(i+1)\}}$ ← "descents"

The $E_n(t)$ compile nicely in an exponential generating function (EGF),

$$\sum_{n \geq 0} E_n(t) \frac{x^n}{n!} = \frac{t-1}{t - e^{x(t-1)}}$$

4. REU Problem #7:

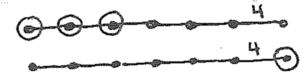
(a) Use Renner's classification of the simple P_λ's in all types, [Renner 2009, Thm 3.2] and continue the work of Golubitsky (2014) by computing the f/h vectors (and compiling them in generating functions) for them as families.

e.g., Type A 

done by Golubitsky

simple

≥ 2

Type B/C:  simple

(b) Free Renner's results from the Weyl group hypothesis, by showing they were already known to Maxwell, Scharlau, ...