REU 2019 Day 4 Vic Reiner

Dihedral Sieving Phenomena

(based on work of S. Rao & J. Suk REU 2017)

1. Cyclic sieving & q-analogues
   - (a) subsets & q-binomials
   - (b) polygon dissections & q-Catalans
2. Dihedral sieving
   - (a) subsets
   - (b) triangulations & (q,t)-Catalans
3. REU Problem 4

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1. Cyclic sieving

Let's have in mind a (finite) set X, e.g. $X = \binom{[n]}{k} :=$ k-element subsets of $[n] := \{1,2,\ldots,n\}$

and some polynomial $X(q) \in \mathbb{Z}[q]$ that is a q-analogue for $\#X$

meaning $\left[X(q)\right]_{q=1} = \#X$

e.g. $\#X = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ above

e.g. $X(q) :=$ q-binomial coefficient $:= \frac{[n]!_q}{[k]!_q\,[n-k]!_q} \overset{q=1}{\rightsquigarrow} \binom{n}{k}$ where $[n]!_q := [n]_q[n-1]_q \cdots [3]_q[2]_q[1]_q \overset{q=1}{\rightsquigarrow} n!$

and $[n]_q = 1+q+q^2+\cdots+q^{n-1} \overset{q=1}{\rightsquigarrow} n$

e.g. $n=6$, $k=3$:
$$\begin{bmatrix}6\\3\end{bmatrix}_q = \frac{[6]_q[5]_q[4]_q\cancel{[3]_q[2]_q[1]_q}}{[3]_q[2]_q[1]_q\cancel{[3]_q[2]_q[1]_q}} = \frac{(1+q+\cdots+q^5)(1+q+\cdots+q^4)(1+q+q^2+q^3)}{(1+q+q^2)(1+q)(1)} = 1+q+2q^2+3q^3+3q^4+3q^5+3q^6+2q^7+q^8+q^9$$

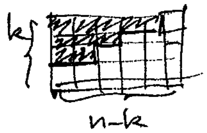
$n=6$, $k=2$:
$$\begin{bmatrix}6\\2\end{bmatrix}_q = \frac{[6]_q[5]_q}{[2]_q[1]_q} = 1+q+2q^2+2q^3+3q^4+2q^5+2q^6+q^7+q^8$$

(2)

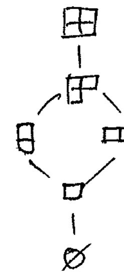
REU Exercise

(a) Show $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ "q-Pascal"

(b) Show $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum\limits_{\substack{\text{Ferrers diagrams} \\ \lambda \subset k \times (n-k)}} q^{\#\text{cells}(\lambda)}$ $\quad (\in \mathbb{N}[q])$



e.g. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4 + q^3 + 2q^2 + q^1 + 1$



(c) Show $\begin{bmatrix} n \\ k \end{bmatrix}_{q=p^d} = \#$ k-dimensional $\mathbb{F}_{p^d}$-linear subspaces of $(\mathbb{F}_{p^d})^n$

for a prime p

Now assume the set X has the action of a cyclic group $C := C_m \cong \mathbb{Z}/m\mathbb{Z}$
$= \langle \rho \rangle = \{1, \rho, \rho^2, \ldots, \rho^{m-1}\}$

permuting it, e.g. $C = C_n \overset{\text{"acts on"}}{\circlearrowright} X = \binom{[n]}{k}$ via $\rho(\{i_1, i_2, \ldots, i_k\}) := \{\underbrace{i_1+1, i_2+1, \ldots, i_k+1}\}$

read these values mod n

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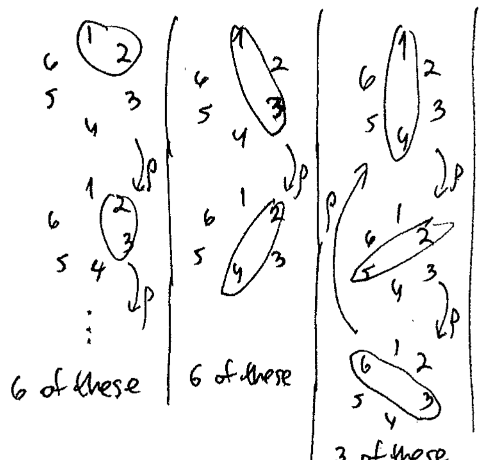
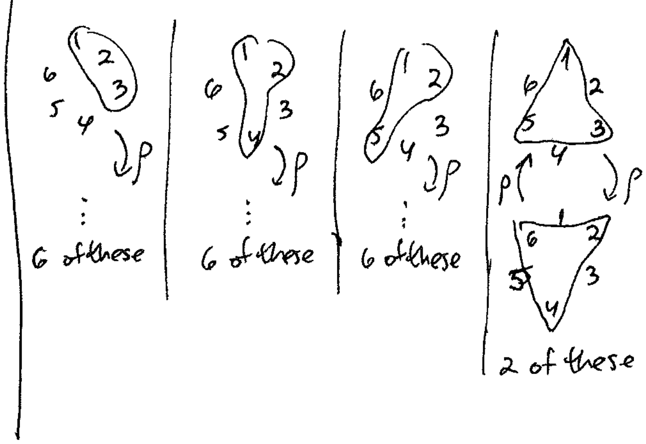
e.g. $X = \binom{[6]}{2}$ has 3 orbits under the C_6-action:



... while $X = \binom{[6]}{3}$ has 4 orbits under the C_6-action:



DEF'N: Say $(X \circlearrowleft C_m, X(q))$ exhibits a <u>cyclic sieving phenomenon</u> (CSP)

(Stanton-White -R. 2004) if for every $\rho^k \in C_m = \langle \rho \rangle$ one has $\#X^{\rho^k} := \#\{x \in X : \rho^k(x) = x\}$

$$= \big[X(q)\big]_{q = \underbrace{\left(e^{2\pi i/m}\right)^k}_{\zeta_k :=}}$$

(a) THM (RSW 2004) $\left(X = \binom{[n]}{k} \circlearrowleft C_n,\ X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q\right)$ exhibits ~~a~~ CSP.

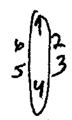
e.g. $\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8$

$q=1$: $\binom{6}{2} = 15 = \#X$ no surprise!

$q = -1 = \zeta_6^3$: $3 = \#X^{\rho^3}$

$q = e^{\frac{2\pi i}{3}} = \zeta_6^2$: $0 = \#X^{\rho^2}$

$q = e^{\frac{2\pi i}{6}} = \zeta_6$: $0 = \#X^{\rho^1}$



e.g. $\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = \cdots\cdots$

$q=1$: $\binom{6}{3} = 20$

$q=-1$: 0

$q = e^{2\pi i/3}$: $2 = \#X^{\rho^2}$

$q = e^{2\pi i/6}$: 0

(b) THM (Kirkman 1857 (CONJ.) - Cayley 1890): $\#\{$dissections of $(n+2)$-sided polygon using k noncrossing diagonals$\} = \frac{1}{n}\binom{n}{k+1}\binom{n+k+1}{k}$

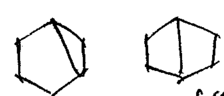
$k = n-1$:

$\#\{$triangulations of $(n+2)$-gon$\} = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n} =:$ Catalan number

e.g. $n = 4$ $k = 3$ $\#\{$triangulations of a 6-gon$\} = \frac{1}{4}\binom{8}{3} = \frac{1}{5}\binom{8}{4} = 14$

6 of these 3 of these 3 of these 2 of these

$k = 1$ $\#\{$dissections of a 6-gon with 1 diagonal$\} = \frac{1}{4}\binom{4}{2}\binom{6}{1} = 9$



6 of these 3 of these

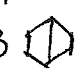
THM (RSW 2004) $\left(X = \left\{\begin{matrix}(n+2)\text{-gon dissections} \\ \text{with } k \text{ diagonals}\end{matrix}\right\} \circlearrowleft C_{n+2},\ X(q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q\right)$ exhibits a CSP.

$k = n-1$: $\frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q$ MacMahon's q-Catalan number

e.g. $n = 4$ $k = 1$ $X(q) = \frac{1}{[4]_q}\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \begin{bmatrix} 6 \\ 1 \end{bmatrix}_q = \frac{\cancel{[4]_q}[3]_q}{\cancel{[4]_q}[2]_q}[6]_q = (1+q+q^2)(1+q^2+q^4)$

$q=1$: 9

$q=-1$: 3



$q = e^{2\pi i/3}$: 0

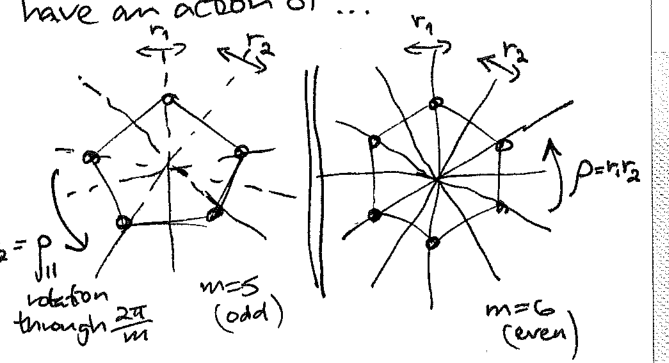
$q = e^{\frac{2\pi i}{6}}$: 0

(4)

2. Dihedral sieving

Lots of combinatorial sets X with C_m-action have an action of ...

$$I_2(m) := \text{dihedral group of size } 2m$$
$$:= \text{(linear) symmetries of (regular) } 2m\text{-gon}$$
$$= \{\underbrace{1, \rho, \rho^2, \ldots, \rho^{m-1}}_{m \text{ rotations}}, \underbrace{r_1, r_1\rho, r_1\rho^2, \ldots, r_1\rho^{m-1}}_{m \text{ reflections}}\}$$



REU Exercise

(a) Show $I_2(m)$ has this presentation: $I_2(m) \cong \langle \underbrace{\overset{r_1}{\overset{\parallel}{r}}, \rho}_{\text{generators}} \mid \underbrace{r^2 = \rho^m = 1,\ r\rho r^{-1} = \rho^{-1}}_{\text{relations}} \rangle$

(b) Show that the defining representation (= group homom. into some $GL(V)$)

$$I_2(m) \xrightarrow{\varphi_{def}} GL_2(\mathbb{R}) \ (\subset GL_2(\mathbb{C}))$$
$$r \longmapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\rho \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \ \theta = 2\pi/m$$

is equivalent via some change-of-basis $P \in GL_2(\mathbb{C})$ to this representation:

meaning $P^{-1}\varphi_{def}(w)P = \varphi^{(1)}(w) \ \forall w \in I_2(m)$

$$I_2(m) \xrightarrow{\varphi^{(1)}} GL_2(\mathbb{C})$$
$$r \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\rho \longmapsto \begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix}$$

(c) Show that all of the inequivalent characters (= homom./representations into $GL_1(\mathbb{C}) = \mathbb{C}^\times$) $I_2(m) \xrightarrow{\chi} GL_1(\mathbb{C}) = \mathbb{C}^\times$

are these:

	$\overset{r}{\overset{\parallel}{r_1}}$	r_2	$\rho = r_1 r_2$
$\mathbb{1}$	+1	+1	+1
χ_1	−1	+1	−1
χ_2	+1	−1	−1
def	−1	−1	+1

only allowed if m even

(d) Show ~~that~~ $I_2(m) \xrightarrow{\varphi^{(k)}} GL_2(\mathbb{C})$ for $k = 1, 2, \ldots, m-1$

$$r \longmapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\rho \longmapsto \begin{bmatrix} \zeta_m^k & 0 \\ 0 & \zeta_m^{-k} \end{bmatrix}$$

define representations of $I_2(m)$ that are inequivalent for $k = 1, 2, \ldots, \lfloor \frac{m-1}{2} \rfloor$

(5)

Q: Is there a notion of <u>dihedral</u> <u>sieving</u> phenomenon (D.S.P.) for a set $X \circlearrowleft I_2(m)$?

A: <u>Yes</u>, at least for <u>m odd</u>!

DEF'N (Rao-Suk 2017) (rephrased) Given a finite set $X \circlearrowleft I_2(m)$ with <u>m odd</u> and a polynomial $X(q,t) \in \mathbb{Z}[q,t]$ that is <u>symmetric in q,t</u> $\left(X(q,t) = X(t,q)\right)$

say $(X \circlearrowleft I_2(m), X(q,t))$ exhibits a D.S.P. if every $w \in I_2(m)$ has $\#X^{w} := \#\{x \in X : w(x) = x\} = \left[X(q,t)\right]_{q=\lambda_1,\ t=\lambda_2}$

where $\{\lambda_1, \lambda_2\}$ are the <u>two eigenvalues</u> of $\varphi_{def}(w)$ or $\varphi^{(1)}(w)$ in $GL_2(\mathbb{C})$

equivalently: $\det(x I_2 - \varphi_{def}(w)) = x^2 - (\lambda_1+\lambda_2)x + \lambda_1\lambda_2$

i.e. $\begin{cases} \#X^{r} = \left[X(q,t)\right]_{q=+1,\ t=-1} \\ \#X^{\rho^k} = \left[X(q,t)\right]_{q=\zeta_m^k,\ t=\zeta_m^{-k}} \text{ for } k=0,1,\dots,m-1 \end{cases}$ and

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(a) <u>Subsets</u>

<u>THM</u> (Rao-Suk 2017) For n odd $\left(X = \binom{[n]}{k} \circlearrowleft I_2(n),\ X(q,t) = \left\{ {n \atop k} \right\}_{q,t}\right)$ exhibits a DSP

where $\left\{ {n \atop k} \right\}_{q,t} := \dfrac{\{n\}!_{q,t}}{\{k\}!_{q,t}\{n-k\}!_{q,t}}$ and $\{n\}!_{q,t} := \{n\}_{q,t}\{n-1\}_{q,t}\cdots\{3\}_{q,t}\{2\}_{q,t}\{1\}_{q,t}$

and $\{n\}_{q,t} := q^{n-1} + q^{n-2}t + q^{n-3}t^2 + \dots + qt^{n-2} + t^{n-1}$

related to Fibonomials of Amdeberhan-Chen-Moll-Sagan

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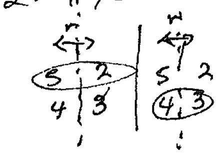
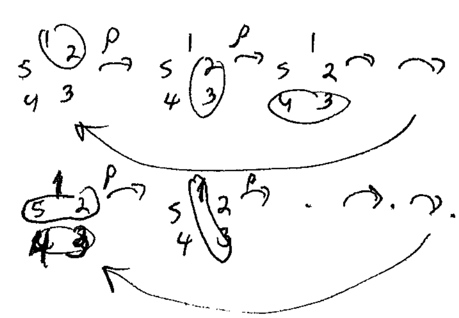
e.g. $n=5$, $k=2$: $X(q,t) = \left\{ {5 \atop 2} \right\}_{q,t} = \dfrac{\{5\}_{q,t}\{4\}_{q,t}}{\{2\}_{q,t}\{1\}_{q,t}} = q^6 + q^5t + 2q^4t^2 + 2q^3t^3 + 2q^2t^4 + qt^5 + t^6 \quad \left(= t^{\overset{2(5-2)\cdot 2}{12}} \left[{5 \atop 2} \right]_{q \mapsto q/t}\right)$

$q=1,\ t=1$: $\binom{5}{2} \overset{\checkmark}{=} \#X$

$q=\zeta_5,\ t=\zeta_5^{-1}$: $0 \overset{\checkmark}{=} \#X^{\rho}$

$q=+1,\ t=-1$: $+1-1+2-2+2-1+1 = +2 \overset{\checkmark}{=} \#X^{r}$





(6)

(b) Triangulations

THM (Rao & Suk 2017) For n odd $\left(X = \left\{ \begin{smallmatrix} \text{triangulations} \\ \text{of } (n+2)\text{-gon} \end{smallmatrix} \right\} \circlearrowleft I_2(n+2),\ X(q,t) = (qt)^{\binom{n}{2}} \mathrm{Cat}_n(q,t) \right)$

exhibits a DSP, where

$\mathrm{Cat}_n(q,t) :=$ <u>Garsia-Haiman (q,t)-Catalan polynomial</u>

$:=$ (q,t)-bigraded Hilbert series of sgn-isotypic component of diagonal harmonics

$$= \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] \Big/ \underbrace{\left(\mathbb{C}[\underline{x},\underline{y}]_+^{S_n}\right)}_{\substack{\text{ideal gen'd by } S_n \text{ symmetric} \\ \text{polynomials of positive degree}}}$$

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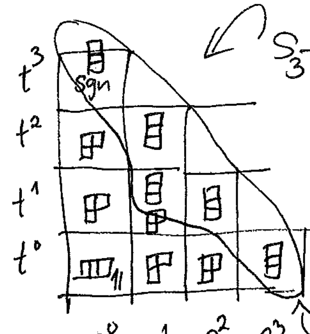
e.g. $n=2$ ~~$\mathbb{C}[x_1,x_2$~~ $\mathbb{C}[x_1,x_2,y_1,y_2] \Big/ \begin{pmatrix} x_1+x_2, & y_1+y_2, \\ x_1^2+x_2^2 & y_1^2+y_2^2, \\ x_1y_1+x_2y_2 & \end{pmatrix}$ this ideal also contains $2x_1^2$, $2y_1^2$, $2x_1y_1$

$$= \begin{array}{c|c|c|} t^1 & y_1 & \\ \hline t^0 & 1 & x_1 \\ \hline & q^0 & q^1 \end{array} \quad = \begin{array}{c|c|c|} t^1 & \mathrm{sgn} & \\ \hline t^0 & 1\!\!1 & \mathrm{sgn} \\ \hline & q^0 & q^1 \end{array} \qquad \mathrm{Cat}_2(q,t) = q^1 + t^1$$

$n=3$ $\mathbb{C}[x_1,x_2,x_3,y_1,y_2,y_3] \Big/ \begin{pmatrix} x_1+x_2+x_3, & y_1+y_2+y_3 \\ x_1^2+\dots & y_1^2+\dots \\ x_1^3+\dots & y_2^2+\dots \\ x_1^2y_1+\dots & \\ x_1y_1^2+\dots & \end{pmatrix}$



S_3-irreducible decomposition from ~~SAGE~~ COCALC or [Haiman 1993]

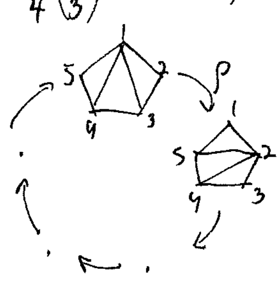
$\mathrm{Cat}_3(q,t) = q^3+q^2t+qt^2+t^3+qt$

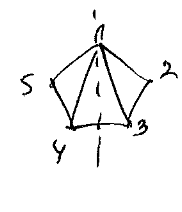
$$(qt)^{\binom{3}{2}} \mathrm{Cat}_3(q,t) = (qt)^3(q^3+q^2t+qt^2+t^3+qt)$$

$q=t=1$: $\frac{1}{4}\binom{6}{3} = 5 = \#X$

$q=\zeta_5$, $t=\zeta_5^{-1}$: $0 = \#X^\rho$

$q=+1$, $t=-1$: $(-1)^3(1-1+1-1-1) = +1 = \#X^r$





3. REU Problem #4

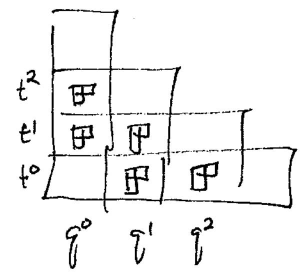
(a) Prove CONJ (Rao–Suk 2017) For n odd, one has a DSP for

$\left(X = \left\{ \begin{array}{l} \text{dissections of} \\ (n+2)\text{-gon using} \\ k \text{ diagonals} \end{array} \right\} \circlearrowright I_2(n+2),\ X(q,t) = \left(\begin{array}{c} \text{"little" } (q,t)\text{-Schröder} \\ \text{polynomial} \end{array} \right)(q,t)^{?} \right)$

$\longrightarrow \tilde{S}_{n,n-1-k}(q,t)$

gives the (bigraded) multiplicities for the S_n-irreducible  in $\mathbb{C}[x,y]/(\mathbb{C}[x,y]^{S_n}_{+})$

e.g. $\tilde{S}_{3,1}(q,t) = q^2 + qt + t^2 + q + t \longrightarrow$



(b) Generalize to (q,t)-Fuss-Catalan's [$X(q,t)=$]

and $X = \left\{ \begin{array}{l} \text{quadrangulations of polygons} \\ \text{pentangulations} \\ \text{etc.} \end{array} \right\}$ ← see work of Eu & Fu on CSP for these.

and to (q,t)-Fuss-Schröder's with dissections

(c) Maybe even rational (q,t)-Catalan's?

(d) Q: What is a DSP for $X \circlearrowright I_2(m)$ with m even??