Chapter 1
Representations of Quivers

In this chapter, we introduce the concept of quiver representations and their morphisms, discuss direct sums, kernels, and cokernels, and study short exact sequences of quiver representations. We also introduce some basic notions of category theory.

1.1 Definitions and Examples

A quiver representation is a finite collection of vector spaces and linear maps between these vector spaces. One can visualize this concept using a diagram of arrows, the quiver, where each arrow represents one of the linear maps.

1.1.1 Representations

In order to study quiver representations we need a formal definition of quivers first.

Definition 1.1. A **quiver** $Q = (Q_0, Q_1, s, t)$ consists of

Q_0 a set of vertices,
Q_1 a set of arrows,
$s\colon Q_1 \to Q_0$ a map from arrows to vertices, mapping an arrow to its starting point,
$t\colon Q_1 \to Q_0$ a map from arrows to vertices, mapping an arrow to its terminal point.

We will represent an element $\alpha \in Q_1$ by drawing an arrow from its starting point $s(\alpha)$ to its endpoint $t(\alpha)$ as follows:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

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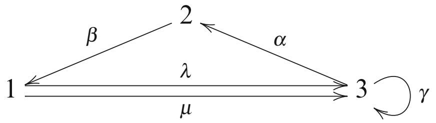
Example 1.1. The following quiver is given by $Q_0 = \{1, 2, 3\}$, $Q_1 = \{\alpha, \beta, \gamma, \lambda, \mu\}$, $s(\alpha) = 3, s(\beta) = 2, s(\gamma) = 3, s(\lambda) = 1, s(\mu) = 1$ and $t(\alpha) = 2, t(\beta) = 1, t(\gamma) = 3, t(\lambda) = 3, t(\mu) = 3$.



A quiver Q is called *finite* if Q_0 and Q_1 are finite sets. We will always suppose our quivers to be finite.

For the definition of quiver representations, we need a field k. For simplicity, we let k be an algebraically closed field.

Definition 1.2. A **representation** $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of a quiver Q is a collection of k-vector spaces

$$M_i$$

one for each vertex $i \in Q_0$, and a collection of k-linear maps

$$\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$$

one for each arrow $\alpha \in Q_1$.

A representation M is called **finite-dimensional** if each vector space M_i is finite-dimensional. In this case the **dimension vector** $\underline{\dim}\, M$ of M is the vector $(\dim M_i)_{i \in Q_0}$ of the dimensions of the vector spaces. An **element** of a representation M is a tuple $(m_i)_{i \in Q_0}$ with $m_i \in M_i$.

Example 1.2. Let Q be the quiver $1 \to 2$. Then

$$\begin{array}{ll}
M & k \xrightarrow{\;\;1\;\;} k \\
M' & k \xrightarrow{\;\;0\;\;} k \\
M'' & k \xrightarrow{\;\;0\;\;} 0 \\
M''' & k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}} k^3
\end{array}$$

are representations of Q. The dimension vectors are $\underline{\dim}\, M = \underline{\dim}\, M' = (1,1)$, $\underline{\dim}\, M'' = (1,0)$, and $\underline{\dim}\, M''' = (2,3)$.

The subject of this book is to study finite-dimensional quiver representations.

1.1.2 Morphisms

Definition 1.3. Let Q be a quiver and let $M = (M_i, \varphi_\alpha)$, $M' = (M'_i, \varphi'_\alpha)$ be two representations of Q. A **morphism** (or homomorphism) of representations $f\colon M \to M'$ is a collection $(f_i)_{i \in Q_0}$ of linear maps

$$f_i\colon M_i \longrightarrow M'_i$$

such that for each arrow $i \overset{\alpha}{\to} j$ in Q_1 the diagram

$$\begin{CD} M_i @>{\varphi_\alpha}>> M_j \\ @V{f_i}VV @VV{f_j}V \\ M'_i @>{\varphi'_\alpha}>> M'_j \end{CD}$$

commutes, that is,

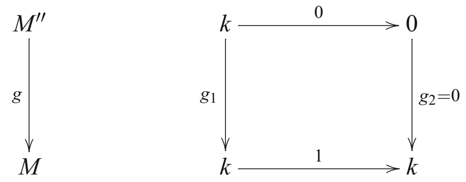
$$f_j \circ \varphi_\alpha(m) = \varphi'_\alpha \circ f_i(m) \quad \text{for all } m \in M_i.$$

A morphism $f = (f_i)\colon M \to N$ is an **isomorphism** if each f_i is bijective. The class of all representations that are isomorphic to a given representation M is called the **isoclass** of M.

Example 1.3. Let us consider the representations in Example 1.2 again. The map $f = (f_1, f_2)$, where f_1 is the multiplication by $a \in k$ and f_2 is the zero map, is a morphism from M to M'':

$$\begin{CD} M \\ @V{f}VV \\ M'' \end{CD} \qquad\qquad \begin{CD} k @>{1}>> k \\ @V{a}VV @VV{0}V \\ k @>{0}>> 0 \end{CD}$$

Now let us see if there are there any morphisms $g\colon M'' \to M$. Suppose we have a commutative diagram:



Then $g_1 = 1 \circ g_1$ must be equal to the zero map, and thus $g_1 = 0$. We have shown that the only morphism from M'' to M is the zero morphism $g = (0, 0)$.

Given a quiver Q, the finite-dimensional representations of Q together with the morphisms of representations form a category which we denote by rep Q.

Categories 1 *We will work with categories throughout the book, and we will develop the language of category theory along the way. For a formal definition see Categories 2 at the end of Sect. 1.2. For now, it suffices to know that a category consists of objects and morphisms.*

We write $M \in \text{rep}\, Q$ if M is an object in rep Q, that is, if M is a finite-dimensional representations of the quiver Q.

Proposition 1.1. *Let $M, M' \in$ rep Q. Then the set of all morphisms Hom(M, M') is a k-vector space with respect to the addition and scaling of morphisms.*

Proof. Exercise. □

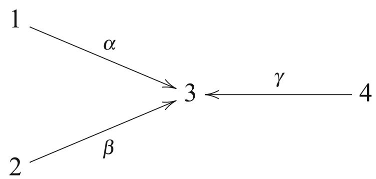
Example 1.4. With the notation of Example 1.3, we have

$$\text{Hom}(M, M'') \cong \{(a, 0) \mid a \in k\} \cong k,$$

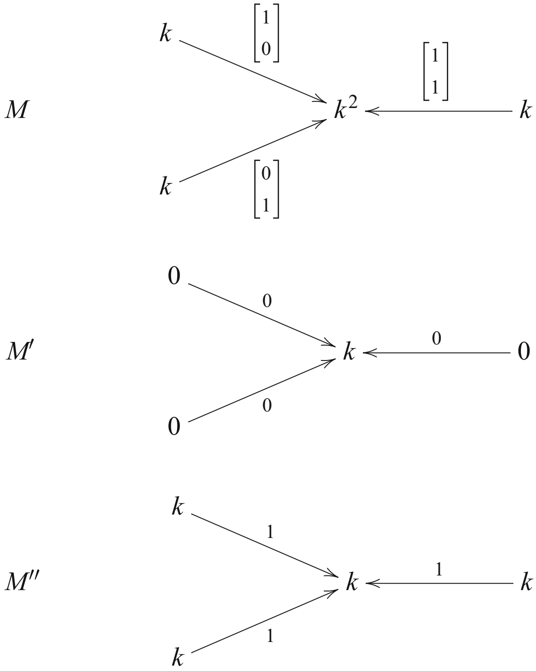
where the last isomorphism holds because the vector space $\{(a, 0) \mid a \in k\}$ is of dimension one. On the other hand,

$$\text{Hom}(M'', M) \cong 0.$$

Example 1.5. Let Q be the quiver



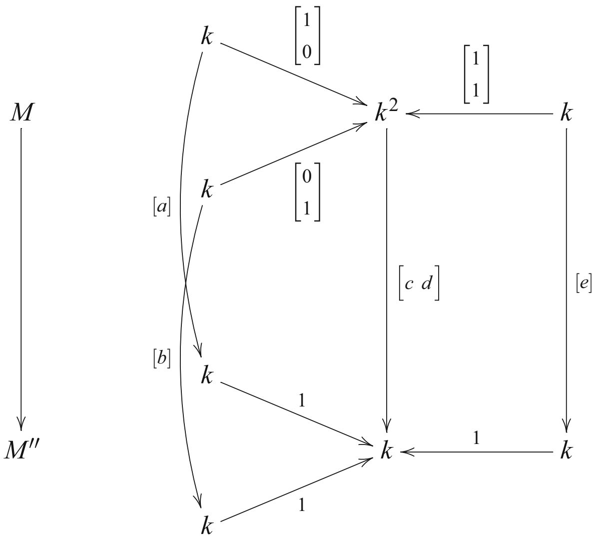
and consider the following representations:



Note that the images of the three maps in M are three different lines in k^2. Then we have

$$\mathrm{Hom}(M, M') = 0 \quad \mathrm{Hom}(M, M'') \cong k^2$$
$$\mathrm{Hom}(M', M) \cong k^2 \quad \mathrm{Hom}(M'', M) = 0.$$

Proof. We show that $\mathrm{Hom}(M, M'') \cong k^2$ and leave the other identities as an exercise. In this example, a morphism $M \to M''$ is a choice of 5 scalars $a, b, c, d, e \in k$ such that the following diagram commutes:



The three commuting squares give the relations

$$a = c, \qquad b = d, \qquad c + d = e.$$

Thus a choice of a and b completely determines the morphism. On the other hand, every choice of a and b yields a different morphism. Therefore $\mathrm{Hom}(M, M'') \cong k^2$.
□

Example 1.6. Let Q be the quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2\ ;$$

This quiver is known as the Kronecker quiver.[1] Consider the following representations of Q:

[1]Leopold Kronecker (1823–1891) studied the problem of classifying pairs of matrices of the same size up to simultaneous conjugation, which is equivalent to studying the representations of the Kronecker quiver. The concept of quivers was introduced much later (1972) by Gabriel [33].

$$M \qquad k^2 \underset{\begin{bmatrix}0\\1\end{bmatrix}}{\overset{\begin{bmatrix}1\\0\end{bmatrix}}{\leftleftarrows}} k$$

$$M' \qquad k^2 \underset{\begin{bmatrix}0&0\\1&0\end{bmatrix}}{\overset{\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\leftleftarrows}} k^2$$

We want to compute $\mathrm{Hom}(M, M')$. Therefore, suppose that $f = (f_1, f_2)$ is a morphism from M to M'. Then f_1 and f_2 can be written in matrix form as

$$f_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad f_2 = \begin{bmatrix} x \\ y \end{bmatrix}$$

where $a, b, c, d, x, y \in k$, and since f is a morphism of representations, we have $f_1\varphi_\alpha = \varphi'_\alpha f_2$ and $f_1\varphi_\beta = \varphi'_\beta f_2$; in other words

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

which implies that

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}.$$

Therefore f is of the form

$$f = \left(\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix}\right),$$

and $\mathrm{Hom}(M, M') \cong k^2$ is a two-dimensional vector space with basis

$$\left\{\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right\}.$$

1.2 Direct Sums and Indecomposable Representations

The direct sum $M \oplus N$ of two representations M and N can be though of as considering both M and N at the same time. If we understand M and N, then we understand their direct sum.

The concept of direct sum is more interesting when we go the other way, that is, given a representation X, we can ask if it is possible to decompose X into a direct sum $X = M \oplus N$, with M and N nonzero. If this is the case, then we can try to decompose the direct summands M and N further and eventually get a decomposition $X = M_1 \oplus M_2 \oplus \cdots \oplus M_t$ in which each of the M_i is indecomposable.

Let Q be a quiver.

Definition 1.4. Let $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ be representations of Q. Then

$$M \oplus M' = \left(M_i \oplus M'_i, \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} \right)_{i \in Q_0, \alpha \in Q_1}$$

is a representation of Q called the **direct sum** of M and M'.

Recursively, we define the direct sum of any finite number of representations $M_1, M_2, \ldots, M_t \in \operatorname{rep} Q$ by

$$M_1 \oplus M_2 \oplus \cdots \oplus M_t = (M_1 \oplus \cdots \oplus M_{t-1}) \oplus M_t.$$

Example 1.7. Let Q be the quiver

$$1 \longrightarrow 2 \longleftarrow 3\ ,$$

and consider the representations

$$M \qquad k \xrightarrow{\ 1\ } k \xleftarrow{\ 0\ } 0;$$

$$M' \qquad k^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} k.$$

Then the direct sum $M \oplus M'$ is the representation

$$k \oplus k^2 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} k \oplus k^2 \xleftarrow{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}} 0 \oplus k;$$

which is isomorphic to

$$k^3 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} k^3 \xleftarrow{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} k\,.$$

Definition 1.5. A representation $M \in \text{rep}\, Q$ is called **indecomposable** if $M \neq 0$ and M cannot be written as a direct sum of two nonzero representations, that is, whenever $M \cong N \oplus L$ with $N, L \in \text{rep}\, Q$, then $N = 0$ or $L = 0$.

Example 1.8. The representations in Examples 1.5 and 1.6 are indecomposable. The representation M in Example 1.7 is indecomposable, but M' is not. M' is isomorphic (but not equal) to

$$(k \xrightarrow{1} k \xleftarrow{1} k) \oplus (k \xrightarrow{1} k \xleftarrow{0} 0).$$

In Example 1.2, the representations M and M'' are indecomposable, and the representations M' and M''' are not.

Goal of Representation Theory
Classify all representations of a given quiver Q and all morphisms between them up to isomorphism.

The following theorem shows that in order to attain this goal, it is sufficient to classify all *indecomposable* representations and morphisms between them.

Theorem 1.2 (Krull–Schmidt Theorem). *Let Q be a quiver and let $M \in$ rep Q. Then*

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_t$$

where the $M_i \in$ rep Q are indecomposable and unique up to order.

Proof. If M is indecomposable, there is nothing to show. If M is not indecomposable, then $M = M' \oplus M''$, where M' and M'' are representations of strictly smaller dimension. By induction, we have $M' \cong M'_1 \oplus M'_2 \oplus \cdots \oplus M'_{t'}$ and $M'' \cong M''_1 \oplus M''_2 \oplus \cdots \oplus M''_{t''}$ with all M'_i, M''_i indecomposable. This shows the existence of the decomposition. For the uniqueness see, for example, [8, I.4.10]. □

We close this section with the definition of a category.

Categories 2 *A* ***category*** $\mathscr{C}$ *consists of objects, morphisms, and a binary operation called the composition of morphisms.*

More precisely, let $\mathscr{C}$ *be a class of objects* $Ob(\mathscr{C})$ *and a class of morphisms* $Hom_{\mathscr{C}}$ *such that each morphism* $f \in Hom_{\mathscr{C}}$ *has a unique* source X *and a unique* target Y *in* $Ob(\mathscr{C})$*. We say that* f *is a morphism from* X *to* Y *and write* $f : X \to Y$*. The class of all morphisms from* X *to* Y *is denoted by* $Hom_{\mathscr{C}}(X, Y)$*.*

Then $\mathscr{C}$ is called a *category* if for every three objects X, Y, Z in $\mathrm{Ob}(\mathscr{C})$, there is a binary operation

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(X,Z) \\ (\quad f \quad , \quad g \quad) & \longmapsto & g \circ f \end{array}$$

called the composition of morphisms that satisfies the following axioms:

1. (associativity) If $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$ are morphisms, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. (identity) For every object X there exists a morphism $1_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ called the identity morphism on X such that for every $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and every $g \in \mathrm{Hom}_{\mathscr{C}}(Z, X)$ we have

$$f \circ 1_X = f \quad \text{and} \quad 1_X \circ g = g.$$

1.3 Kernels, Cokernels, and Exact Sequences

Recall from linear algebra that if $f : V \to V'$ is a linear map, then its kernel $\ker f = \{v \in V \mid f(v) = 0\}$ is a subspace of V, and its cokernel $\operatorname{coker} f = V'/\operatorname{im} f = \{v' + f(V) \mid v' \in V'\}$ is a quotient space of V'.

In this section, we will generalize these concepts to representations.

Let Q be a quiver, and let $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $M' = (M'_i, \varphi'_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be two representations of Q. Furthermore, let $f = (f_i)_{i \in Q_0} : M \to M'$ be a morphism of representations. Recall that each f_i is a linear map from the vector space M_i to the vector space M'_i.

For each vertex $i \in Q_0$, let $L_i = \ker f_i$, and for each arrow $i \xrightarrow{\alpha} j$ in Q_1, let $\psi_\alpha : L_i \to L_j$ be the restriction of φ_α to L_i, that is, $\psi_\alpha(x) = \varphi_\alpha(x)$ for all $x \in L_i$. Let us check that ψ_α is well defined. We must show that for all $x \in L_i$, we have $\psi_\alpha(x) \in L_j$ which means that $\varphi_\alpha(x) \in \ker f_j$. But since f is a morphism of representations, we have $f_j \varphi_\alpha(x) = \varphi'_\alpha f_i(x)$, which is zero, since $x \in \ker f_i$. This shows that ψ_α is well defined.

Definition 1.6. The representation $\ker f = (L_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is called the **kernel** of f.

Remark 1.3. The inclusions $\mathrm{incl}_i : \ker f_i \hookrightarrow M_i$ induce an injective morphism of representations[2]:

$$(\mathrm{incl}_i)_{i \in Q_0} : \ker f \hookrightarrow M.$$

Next we define the cokernel of the morphism f. For each vertex $i \in Q_0$, let $N_i = \operatorname{coker} f_i = M'_i / f_i(M_i)$, and for each arrow $i \xrightarrow{\alpha} j$ in Q_1, define $\chi_\alpha : N_i \to N_j$ by

$$\chi_\alpha(m'_i + f_i(M_i)) = \varphi'_\alpha(m'_i) + f_j(M_j),$$

for each $m'_i \in M'_i$.

Let us check that χ_α is well defined. Suppose we have two elements $m'_i, m''_i \in M'_i$ such that $m'_i + f_i(M_i) = m''_i + f_i(M_i)$. Then $m'_i - m''_i \in f(M_i)$ and thus $\varphi'_\alpha(m'_i) - \varphi'_\alpha(m''_i) = \varphi'_\alpha(m'_i - m''_i)$ lies in $\varphi'_\alpha f_i(M_i) = f_j \varphi_\alpha(M_i) \subset f_j(M_j)$. It follows that $\chi_\alpha(m'_i + f_i(M_i)) = \chi_\alpha(m''_i + f_i(M_i))$, and therefore χ_α is well defined.

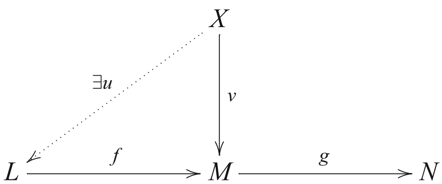
Definition 1.7. The representation $\operatorname{coker} f = (N_i, \chi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is called the **cokernel** of f.

Remark 1.4. The projections $\mathrm{proj}_i : M'_i \twoheadrightarrow \operatorname{coker} f_i$ induce a surjective morphism of representations[3]:

$$(\mathrm{proj}_i)_{i \in Q_0} : M' \twoheadrightarrow \operatorname{coker} f.$$

In category theory, kernels and cokernels are defined using the following *universal properties*:

Remark 1.5. Let $M \xrightarrow{g} N$ be a morphism. Then a kernel of g is a morphism $L \xrightarrow{f} M$ such that $gf = 0$, and given any morphism $X \xrightarrow{v} M$ such that $gv = 0$, there is a unique morphism $X \xrightarrow{u} L$ such that $fu = v$. We say that v factors through f:



[2] The arrow $\hookrightarrow$ indicates that the morphism is injective.

[3] The arrow $\twoheadrightarrow$ indicates that the morphism is surjective.

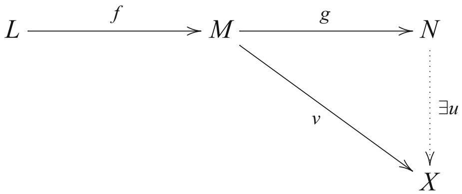
Let us prove that our kernel from Definition 1.6 satisfies this universal property. So suppose that $g : M \to N$ is a morphism of representations of a quiver Q, let $L = \ker g$ as in Definition 1.6 and let f be the inclusion map $f : L \hookrightarrow M$. Then for every vertex $i \in Q_0$ and every $m_i \in L_i$, we have $g_i f_i(m_i) = g_i(m_i) = 0$, which shows that $gf = 0$.

Now suppose that $v : X \to M$ is a morphism of representations such that $gv = 0$. Let us use the notation $M = (M_i, \varphi_\alpha)$, $L = (L_i, \psi_\alpha)$, and $X = (X_i, \chi_\alpha)$ for the three representations. Then for every $i \in Q_0$ and every $x_i \in X_i$, we have $v(x_i) \in \ker g_i = L_i$, so we can define a map $u : X \to L$ by $u_i(x_i) = v_i(x_i)$. It is clear that $fu = v$ so that the above diagram commutes, but we must check that u is actually a morphism of representations. So let $i \xrightarrow{\alpha} j$ be an arrow in Q, and let $x_i \in X_i$. Then, using the definitions of u and L and the fact that v is a morphism of representations, we get

$$\psi_\alpha u_i(x_i) = \varphi_\alpha v_i(x_i) = v_j \chi_\alpha(x_i) = u_j \chi_\alpha(x_i),$$

which shows that u is a morphism of representations. The fact that u is unique follows directly from the fact that f is the inclusion morphism.

Remark 1.6. Let $L \xrightarrow{f} M$ be a morphism. Then a cokernel of f is a morphism $M \xrightarrow{g} N$ such that $gf = 0$, and given any morphism $M \xrightarrow{v} X$ such that $vf = 0$, there is a unique morphism $N \xrightarrow{u} X$ such that $ug = v$. We say that v factors through g:



We leave it to the reader to prove that the two definitions of the cokernel agree.

Definition 1.8. A representation L is called a *subrepresentation* of a representation M if there is an injective morphism $i : L \hookrightarrow M$. In this situation, the *quotient representation* M/L is defined to be the cokernel of i.

Theorem 1.7 (First Isomorphism Theorem). *If $f : M \to N$ is a morphism of representations, then*

$$\operatorname{im} f \cong M/\ker f.$$

Proof. Let $M = (M_i, \varphi_\alpha)$. Then $\operatorname{im} f$ is the representation $\operatorname{im} f = (f(M_i), \psi_\alpha)$ whose maps are defined by $\psi_\alpha(f_i(m_i)) = f_j \varphi_\alpha(m_i)$ for every arrow $i \xrightarrow{\alpha} j$ in Q.

On the other hand, $M/\ker f$ is the representation $M/\ker f = (M_i/\ker f_i, \chi_\alpha)$ where $\chi_\alpha(m_i + \ker f_i) = \varphi_\alpha(m_i) + \ker f_j$.

Since each f_i is a linear map, it induces an isomorphism of vector spaces:

$$\bar{f_i} : M_i/\ker f_i \to f_i(M_i),\ m_i + \ker f_i \mapsto f_i(m_i).$$

Moreover, for every arrow $i \xrightarrow{\alpha} j$, we have $\psi_\alpha \bar{f_i} = \bar{f_j}\varphi_\alpha$, which shows that $\bar{f}$ is a morphism of representations. This completes the proof. □

Categories 3 *With the above definition of kernel and cokernel, we have that rep Q is an* abelian k-category. *This means that*

1. *rep Q is a k*-category, *that is, Hom(M, N) is a k-vector space for all $M, N \in$ rep Q, and the composition of morphisms is bilinear,*
2. *rep Q is* additive, *that is, rep Q has direct sums, there is a zero object $0 \in$ rep Q such that the identity morphism $1_0 \in$ Hom$(0,0)$ is the zero of the vector space Hom$(0,0)$, and*
3. *each morphism $f : M \to N$ in rep Q has a kernel $i : K \to M$ and a cokernel $p : N \to C$ such that the cokernel of i is isomorphic to the kernel of p.*

Observe that the condition coker $i \cong$ $\ker p$ *in 3 follows from the first isomorphism theorem.*

Next we introduce the notion of exact sequences which will be fundamental for the rest of the book.

Definition 1.9. A sequence of morphisms $L \xrightarrow{f} M \xrightarrow{g} N$ is called *exact at M* if $\operatorname{im} f = \ker g$. A sequence of morphisms

$$\cdots \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \cdots$$

is called *exact* if it is exact at every M_i.

Definition 1.10. A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0\ .$$

Note that the sequence in Definition 1.10 is short exact if and only if f is injective, $\operatorname{im} f = \ker g$, and g is surjective.

Example 1.9. Let $f : M \to N$ be a morphism in rep Q. Then the sequence

$$0 \longrightarrow \ker f \xrightarrow{u} M \xrightarrow{f} N \xrightarrow{p} \operatorname{coker} f \longrightarrow 0\ ,$$

where u is the inclusion of Remark 1.3 and p, the projection of Remark 1.4, is exact; and the sequence

$$0 \longrightarrow \ker f \xrightarrow{u} M \xrightarrow{q} M/\ker f \longrightarrow 0$$

is short exact.

Example 1.10. Let Q be the quiver $1 \longrightarrow 2$, and consider the three representations:

$$\begin{array}{ll} S(2) & (0 \longrightarrow k), \\ M & (k \xrightarrow{1} k), \\ S(1) & (k \longrightarrow 0). \end{array}$$

Then

$$\begin{array}{l} 0 \longrightarrow S(2) \xrightarrow{f} M \xrightarrow{g} S(1) \longrightarrow 0 \\ 0 \longrightarrow S(2) \xrightarrow{f'} S(1) \oplus S(2) \xrightarrow{g'} S(1) \longrightarrow 0, \end{array}$$

where $f = (f_1, f_2) = (0, 1)$, $g = (g_1, g_2) = (1, 0)$, and $f' = (f_1', f_2') = (0, 1)$, $g' = (g_1', g_2') = (1, 0)$ are short exact sequences.

Definition 1.11. A morphism $f : L \to M$ is a called a **section** if there exists a morphism $h : M \to L$ such that $h \circ f = 1_L$.

A morphism $g : M \to N$ is a called a **retraction** if there exists a morphism $h : N \to M$ such that $g \circ h = 1_N$.

Definition 1.12. We say that a short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

splits if f is a section.

Example 1.11. In Example 1.10, the second short exact sequence splits, because the morphism $h' : S(1) \oplus S(2) \xrightarrow{(0,1)} S(2)$ verifies $h' \circ f' = 1_{S(2)}$. On the other hand, the first sequence does not split, since there is no nonzero morphism from M to $S(2)$; hence f cannot be a section.

Proposition 1.8. *Let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence in rep Q. Then

(a) *f is a section if and only if g is a retraction.*
(b) *If f is a section, then im $f (= \ker g)$ is a direct summand of M.*

Proof. First we show (a).

($\Rightarrow$) Suppose that f is a section. Then there exists $h \in \mathrm{Hom}(M, L)$ such that $h \circ f = 1_L$.

Define $h' : N \to M$ as follows: Let $n \in N$. Since g is surjective, there exists $m \in M$ such that $g(m) = n$. Choose one such m and define $h'(n) = m - f \circ h(m)$.

Since there may be different m to choose from, we must show that the definition of h' does not depend on the choice of m. Suppose that $m, m' \in M$ are such that $g(m) = g(m') = n$. We must show that $m - f \circ h(m) = m' - f \circ h(m')$. We have $g(m - m') = g(m) - g(m') = 0$, which shows that $m - m' \in \ker g$, and thus $m - m' \in \mathrm{im}\, f$. Therefore, there exists an $\ell \in L$ such that $f(\ell) = m - m'$, and consequently,

$$\begin{aligned} m - fh(m) - (m' - fh(m')) &= m - m' - fh(m - m') \\ &= m - m' - fhf(\ell) \\ &\overset{(*)}{=} m - m' - f(\ell) \\ &= 0, \end{aligned}$$

where the equation $(*)$ holds, because $h \circ f = 1_L$. This shows that h' is well defined.

Next we show that h' is a morphism. To do so, we need some notation. Let $L = (L_i, \varphi_\alpha)$, $M = (M_i, \varphi'_\alpha)$, and $N = (N_i, \varphi''_\alpha)$. Let $i \xrightarrow{\alpha} j$ be an arrow in Q_1. Then we have the following diagram:

$$\begin{array}{ccc} L_i & \xrightarrow{\varphi_\alpha} & L_j \\ h_i \uparrow\downarrow f_i & & f_j \downarrow\uparrow h_j \\ M_i & \xrightarrow{\varphi'_\alpha} & M_j \\ h'_i \uparrow\downarrow g_i & & g_j \downarrow\uparrow h'_j \\ N_i & \xrightarrow{\varphi''_\alpha} & N_j \end{array}$$

which is commutative with respect to the morphisms f, g and h, and we must show its commutativity with respect to h'. Let $n_i \in N_i$, and let $m_i \in M_i$ such that $g_i(m_i) = n_i$ as in the definition of h'. Then

$$\varphi'_\alpha h'_i(n_i) = \varphi'_\alpha(m_i - f_i h_i(m_i)) = \varphi'_\alpha(m_i) - \varphi'_\alpha f_i h_i(m_i),$$

which is shown to be equal to $\varphi'_\alpha(m_i) - f_j h_j \varphi'_\alpha(m_i)$ by using the commutativity of the diagram first for f and then for h.

On the other hand,

$$h'_j \varphi''_\alpha(n_i) = h'_j \varphi''_\alpha g_i(m_i) = h'_j g_j \varphi'_\alpha(m_i)$$

which is also equal to $\varphi'_\alpha(m_i) - f_j h_j \varphi'_\alpha(m_i)$, by definition of h'. This shows that $\varphi'_\alpha h'_i = h'_j \varphi''_\alpha$ and that h' is a morphism in rep Q.

Finally, we show that $gh' = 1_N$ and, consequently, that g is a retraction. To do so, let $n \in N$ and $m \in M$ such that $g(m) = n$. Then

$$gh'(n) = g(m - fh(m)) = g(m) - gf(h(m)) = g(m),$$

where the last equation holds, since $gf = 0$, because the sequence in the proposition is exact. But $g(m) = n$, and thus $gh' = 1_N$.

($\Leftarrow$) Suppose that g is a retraction. Then there is $h' \in \mathrm{Hom}(N, M)$ such that $g \circ h' = 1_N$. Define $h : M \to L$ as follows. Let $m \in M$, and then $m - h'(g(m)) \in \ker g = \mathrm{im}\, f$. Therefore there is $\ell \in L$ such that $f(\ell) = m - h'g(m)$, and this ℓ is unique, since f is injective. Define $h(m) = \ell$.

Clearly, $h \circ f = 1_L$.

In order to finish the proof, let us check that h is a morphism in rep Q. We will use the same notation as in the first part of the proof. Let $m_i \in M_i$ and let $\ell_i \in L_i$ such that

$$f_i(\ell_i) = m_i - h'_i g_i(m_i) \tag{1.1}$$

as in the definition of h. Then $\varphi_\alpha h_i(m_i) = \varphi_\alpha(\ell_i)$. On the other hand, by definition of h, we have $h_j \varphi'_\alpha(m_i) = \ell_j$ for a unique $\ell_j \in L_j$ with the property that

$$f_j(\ell_j) = \varphi'_\alpha(m_i) - h'_j g_j \varphi'_\alpha(m_i). \tag{1.2}$$

We must show that $\varphi_\alpha h_i = h_j \varphi'_\alpha$, so it suffices to show that $\varphi_\alpha(\ell_i) = \ell_j$. Now

$$\varphi'_\alpha(m_i) = \varphi'_\alpha f_i(\ell_i) + \varphi'_\alpha h'_i g_i(m_i) = f_j \varphi_\alpha(\ell_i) + h'_j g_j \varphi'_\alpha(m_i),$$

where the first identity holds because of (1.1) and the last identity holds because f, h and g are morphisms. Using this last identity to replace the first term on the right hand side of (1.2), we get

$$f_j(\ell_j) = f_j\varphi_\alpha(\ell_i),$$

and since f_j is injective, this implies that $\varphi_\alpha(\ell_i) = \varphi'_\alpha(m_i)$. This proves (a).

In order to prove (b), let $h' \in \operatorname{Hom}(N, M)$ be such that $gh' = 1_N$ and let $m = (m_i)_{i\in Q_0} \in M$. Then $m_i = h'_i g_i(m_i) + (m_i - h'_i g_i(m_i))$ with $h'_i g_i(m_i) \in \operatorname{im} h'_i$ and $(m_i - h'_i g_i(m_i)) \in \ker g_i$. Moreover, $\operatorname{im} h'_i \cap \ker g_i = \{0\}$, since $g \circ h' = 1_N$. Thus for each of the vector spaces M_i, we have a direct sum decomposition $M_i = \operatorname{im} h'_i \oplus \ker g_i$.

We still have to check that the maps of the representation M are the maps of the direct sum $\operatorname{im} h' \oplus \ker g$, that is, we must show that for each arrow $i \xrightarrow{\alpha} j$ in Q_1 we have

$$\varphi'_\alpha = \begin{bmatrix} \varphi'_\alpha|_{\operatorname{im} h'_i} & 0 \\ 0 & \varphi'_\alpha|_{\ker g_i} \end{bmatrix}. \tag{1.3}$$

If $m_i \in \ker g_i$, then $0 = \varphi''_\alpha g_i(m_i) = g_j\varphi'_\alpha(m_i)$, because g is a morphism. Therefore $\varphi'_\alpha(m_i) \in \ker g_j$ and thus the upper right block of the matrix in (1.3) is zero. If $m_i \in \operatorname{im} h'_i$, then there exists $n_i \in N_i$ such that $h'_i(n_i) = m_i$, and therefore $\varphi'_\alpha(m_i) = \varphi'_\alpha h'_i(n_i) = h'_j\varphi''_\alpha(n_i)$ is an element of $\operatorname{im}(h'_j)$. This shows that the lower left block of the matrix in (1.3) is zero, and therefore φ'_α is of the form (1.3), and we are done. □

Corollary 1.9. *If the sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is split exact, then

$$M \cong L \oplus N.$$

Proof. Since f is injective, we have $L \cong f(L) \cong \ker g$, and, since g is surjective, the first isomorphism theorem implies $N \cong M/\ker g$. Now the result follows from Proposition 1.8. □

1.4 Hom Functors

We now want to introduce the Hom functors and study their effect on short exact sequences. First, let us recall the definition of functors.

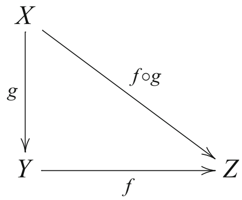
Categories 4 *Let $\mathscr{C}, \mathscr{C}'$ be two k-categories. A* ***covariant functor*** $F : \mathscr{C} \to \mathscr{C}'$ *is a mapping that associates*

- *to each object $X \in \mathscr{C}$ an object $F(X) \in \mathscr{C}'$ and*
- *to each morphism $f : X \to Y$ in $\mathscr{C}$ a morphism $F(f) \colon F(X) \to F(Y)$ in $\mathscr{C}'$,*

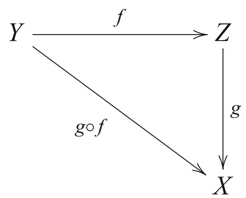
such that $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$, for all objects X and all morphisms f and g in $\mathscr{C}$.
A **contravariant functor** $F : \mathscr{C} \to \mathscr{C}'$ is a mapping that associates:

- to each object $X \in \mathscr{C}$ an object $F(X) \in \mathscr{C}'$ and
- to each morphism $f \colon X \to Y$ in $\mathscr{C}$ a morphism $F(f) \colon F(Y) \to F(X)$ in $\mathscr{C}'$,

such that $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(f) \circ F(g)$, for all objects X and all morphisms f and g in $\mathscr{C}$.
Two very important functors are the Hom functors $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$, where X is an arbitrary fixed object in the category $\mathscr{C}$. They are defined as follows:
$\mathrm{Hom}(X, -)$ is the covariant functor from the category $\mathscr{C}$ to the category of k-vector spaces, which sends an object Y in $\mathscr{C}$ to the vector space $\mathrm{Hom}(X, Y)$ of all morphisms from X to Y and which sends a morphism $(f \colon Y \to Z)$ in $\mathscr{C}$ to the map $f_* \colon \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(X, Z)$, $f_*(g) = f \circ g$:



The map f_* is called the *push forward* of f.
$\mathrm{Hom}(-, X)$ is the contravariant functor from the category $\mathscr{C}$ to the category of k-vector spaces, which sends an object Y in $\mathscr{C}$ to the vector space $\mathrm{Hom}(Y, X)$ of all morphisms from Y to X and which sends a morphism $(f : Y \to Z)$ in $\mathscr{C}$ to the map $f_* \colon \mathrm{Hom}(Z, X) \longrightarrow \mathrm{Hom}(Y, X)$, $f^*(g) = g \circ f$:



The map f^* is called the *pull back* of f.

Let us go back to our category rep Q of representations of the quiver Q. It turns out that applying the Hom functors to short exact sequences of representations yields new exact sequences of vector spaces.

Theorem 1.10. *Let Q be a quiver and $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$ a sequence in rep Q. Then this sequence is exact if and only if for every representation $X \in$ rep Q, the sequence*

$$0 \longrightarrow \mathrm{Hom}(X,L) \xrightarrow{f_*} \mathrm{Hom}(X,M) \xrightarrow{g_*} \mathrm{Hom}(X,N)$$

is exact.

Proof.

($\Rightarrow$) First, we show that f_* is injective. Suppose that there is $u \in \mathrm{Hom}(X, L)$ such that $0 = f_*(u) = f \circ u$. Since f is injective, we can conclude that $u = 0$, and thus f_* is injective.

Next, we show that im $f_* = \ker g_*$. Let $u \in \mathrm{Hom}(X, L)$. Then $g_* f_*(u) = g \circ f \circ u$, which is zero because $g \circ f = 0$. Hence $g_* f_* = 0$, and thus im $f_* \subset \ker g_*$. On the other hand, let $v \in \mathrm{Hom}(X, M)$ such that $v \in \ker g_*$. Then $0 = g_*(v) = g \circ v$. Using the universal property of the kernel of g (Remark 1.5) and the exactness of the sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$, this implies that v factors through f; thus there exists $u \in \mathrm{Hom}(X, L)$ such that $v = f \circ u = f_*(u)$. Thus $v \in$ im f_*, and we have $\ker g_* \subset$ im f_*. Together with the other inclusion above, this implies $\ker g_* =$ im f_*.

($\Leftarrow$) First, we show that f is injective. Take $X = \ker f$, and let $i : X \hookrightarrow L$ be the inclusion morphism. Then $0 = f \circ i = f_*(i)$, and, since f_* is injective, this implies that $i = 0$. But since i is injective, it follows that $X = 0$, and thus f is injective.

Next, we show that im $f = \ker g$. Take $X = L$. Then $0 = g_* f_*(1_L) = g \circ f \circ 1_L = g \circ f$, and thus im $f \subset \ker g$. On the other hand, take $X = \ker g$ and $i : X \hookrightarrow M$ the inclusion morphism. Then $0 = g \circ i = g_*(i)$ implies that $i \in \ker g_* =$ im f_*, and therefore there exists $u \in \mathrm{Hom}(X, L)$ such that $i = f_*(u) = f \circ u$. Consequently $\ker g = i(X) \subset$ im f. Together with the other inclusion above, this implies $\ker g =$ im f, and we are done. □

Corollary 1.11. *A sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \tag{1.4}$$

in rep Q is split exact if and only if for every $X \in$ rep Q, the sequence

$$0 \longrightarrow \mathrm{Hom}(X,L) \xrightarrow{f_*} \mathrm{Hom}(X,M) \xrightarrow{g_*} \mathrm{Hom}(X,N) \longrightarrow 0 \tag{1.5}$$

is exact.

Proof.

($\Rightarrow$) By Theorem 1.10, it suffices to show that g_* is surjective. Suppose that the sequence (1.4) is split exact. Then g is a retraction; hence there is $h \in \mathrm{Hom}(N, M)$ such that $gh = 1_N$.

Now for any $u \in \mathrm{Hom}(X, N)$, we have $hu \in \mathrm{Hom}(X, M)$ and $g_*(hu) = ghu = 1_N u = u$, which shows that g_* is surjective.

($\Leftarrow$) Suppose that for every $X \in$ rep Q, the sequence (1.5) is exact. Then it follows from Theorem 1.10 that the sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N,$$

is exact. Taking $X = N$ and using the surjectivity of g_*, we see that there exists $h \in \mathrm{Hom}(N, M)$ such that

$$1_N = g_*(h) = gh,$$

which proves two facts:

1. g is surjective, which shows that the sequence (1.4) is exact, and
2. g is a retraction, which shows that the sequence (1.4) splits.

□

Remark 1.12. If

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

splits, then

$$0 \longrightarrow \mathrm{Hom}(X,L) \xrightarrow{f_*} \mathrm{Hom}(X,M) \xrightarrow{g_*} \mathrm{Hom}(X,N) \longrightarrow 0$$

splits too. Indeed, $gh = 1_N \Rightarrow g_* h_* = 1_{\mathrm{Hom}(X,N)}$.

There are *dual* versions of Theorem 1.10 and Corollary 1.11 involving the functor $\mathrm{Hom}(-, X)$. We state these results below, but leave the proofs as an exercise. Note that the order of the representations L, M, N is reversed in the Hom sequence, since $\mathrm{Hom}(-, X)$ is contravariant.

Theorem 1.13. *Let Q be a quiver and $L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ a sequence in rep Q. Then this sequence is exact if and only if for every representation $X \in$ rep Q, the sequence*

$$0 \longrightarrow \mathrm{Hom}(N,X) \xrightarrow{g^*} \mathrm{Hom}(M,X) \xrightarrow{f^*} \mathrm{Hom}(L,X)$$

is exact.

Corollary 1.14. *A sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

in rep Q is split exact if and only if for every $X \in$ rep Q, the sequence

$$0 \longrightarrow \mathrm{Hom}(N,X) \xrightarrow{g^*} \mathrm{Hom}(M,X) \xrightarrow{f^*} \mathrm{Hom}(L,X) \longrightarrow 0$$

is exact.

Remark 1.15. If $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ does not split, then f^* and g_* are not always surjective; see the example below. Nevertheless, one can extend the exact sequences of Theorems 1.10 and 1.13 to the right by introducing the *extension functors* $\mathrm{Ext}^i(X,-)$ and $\mathrm{Ext}^i(-,X)$; see Sect. 2.4.

Example 1.12. In Example 1.10 the short exact sequence

$$0 \longrightarrow S(2) \xrightarrow{f} M \xrightarrow{g} S(1) \longrightarrow 0$$

is non-split. Taking $X = S(1)$, and applying $\mathrm{Hom}(S(1),-)$, we get a morphism $g_*\colon \mathrm{Hom}(S(1),M) \to \mathrm{Hom}(S(1),S(1))$ which is not surjective since $\mathrm{Hom}(S(1),M) = 0$ and $\mathrm{Hom}(S(1),S(1)) \cong k$.

1.5 First Examples of Auslander–Reiten Quivers

We have already mentioned that the goal of representation theory of quivers is to study representations and morphisms in rep Q for a given quiver Q. To be even more ambitious, we may add the study of exact sequences in rep Q. In general, the so-called *Auslander–Reiten quiver* is a good first approximation of rep Q. In the case where the number of isoclasses of indecomposable representations is finite, the Auslander–Reiten quiver even provides complete information about rep Q.

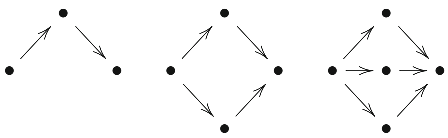


Fig. 1.1 Three different types of meshes

In this section, we give a sneak preview of Auslander–Reiten quivers. More examples will follow in Chap. 3, and for a more rigorous treatment see Chap. 7.

Let Q be a quiver. The Auslander–Reiten quiver of Q is a new quiver Γ_Q whose vertices are the isoclasses of indecomposable representations and whose arrows are given the by so-called *irreducible* morphisms. Roughly speaking, an irreducible morphism between two indecomposable representations is a morphism that does not factor nontrivially through another representation.

Recall that we can build any representation out of indecomposable ones; thus the vertices of the Auslander–Reiten quiver represent the building blocks for the representations.

The arrows of the Auslander–Reiten quiver, the irreducible morphisms, can be thought of the building blocks for morphisms in the sense that many (but in general not all!) morphisms are compositions of irreducible morphisms.

We also want to study short exact sequences of representations. As with morphisms, many of them (but in general not all!) are obtained by gluing together the so-called *almost split sequences*[4]. These almost split sequences are represented in the Auslander–Reiten quiver as meshes; see Fig. 1.1.

Example 1.13. Let Q be the quiver $1 \longrightarrow 2$. It follows from Exercise 1.4 that there are precisely three indecomposable representations (up to isomorphism), namely

$$\begin{array}{ccc} S(2) & M & S(1) \\ 0 \longrightarrow k & k \xrightarrow{\;1\;} k & k \longrightarrow 0. \end{array}$$

We have seen in Example 1.3 that

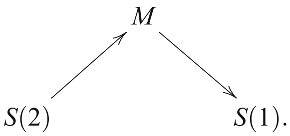
$$\begin{gathered} \mathrm{Hom}(S(1), M) = 0, \;\; \mathrm{Hom}(M, S(2)) = 0, \;\; \mathrm{Hom}(S(2), M) \cong k \\ \mathrm{Hom}(S(1), S(2)) = 0, \, \mathrm{Hom}(M, S(1)) \cong k, \, \mathrm{Hom}(S(2), S(1)) = 0, \end{gathered}$$

and we conclude that there is only one non-split short exact sequence with indecomposable representations at the endpoints:

[4]Maurice Auslander and Idun Reiten introduced the concept of almost split sequences in [10].

$$0 \longrightarrow S(2) \longrightarrow M \longrightarrow S(1) \longrightarrow 0.$$

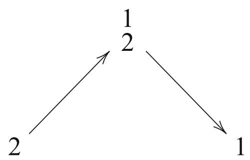
This sequence is actually an almost split sequence. Thus the Auslander–Reiten quiver consists of three vertices, two arrows, and one mesh and is of the form



Remark 1.1. It is often convenient to have a shorthand notation for the representations, encoding the dimension vector and the maps. We will use the following notation throughout the whole book.

Let $Q_0 = \{1, 2, \ldots, n\}$ be the set of vertices of the quiver, let $M = (M_i, \varphi_\alpha)$ be an indecomposable representation of Q, and let $\underline{\dim}\, M = (d_1, d_2, \ldots, d_n)$ be its dimension vector. We describe the representation M as a configuration of digits using $1, 2, \ldots, n$ in such a way that the digit i appears exactly d_i times. Moreover, we arrange the digits in such a way that if there is an arrow $\alpha : i \to j$ such that the corresponding map $\varphi_\alpha : M_i \to M_j$ is nonzero, then the digit i is placed above the digit j. This notation has its limitations, but it is particularly useful if the isomorphism class of the representation M is determined by its dimension vector.

In the example above, we can picture the representation M by ${}^{2}_{1}$, meaning that $M_1 = k$ and $M_2 = k$ and the arrow is going downward from 1 to 2 and carries the identity map. In this notation, the whole Auslander–Reiten quiver would be



Example 1.14. Let Q be the quiver $1 \to 2 \leftarrow 3$. In this case, there are precisely six isoclasses of indecomposable representations, namely

$S(2)$	$P(1)$	$P(3)$
$0 \longrightarrow k \longleftarrow 0$	$k \xrightarrow{1} k \longleftarrow 0$	$0 \longrightarrow k \xleftarrow{1} k$
$I(2)$	$S(1)$	$S(3)$
$k \xrightarrow{1} k \xleftarrow{1} k$	$k \longrightarrow 0 \longleftarrow 0$	$0 \longrightarrow 0 \longleftarrow k$

or, using our symbolic notation,

$$S(2) = 2,\ P(1) = \begin{smallmatrix}1\\2\end{smallmatrix},\ P(3) = \begin{smallmatrix}3\\2\end{smallmatrix},\ I(2) = \begin{smallmatrix}1\ 3\\2\end{smallmatrix},\ S(1) = 1,\ S(3) = 3.$$

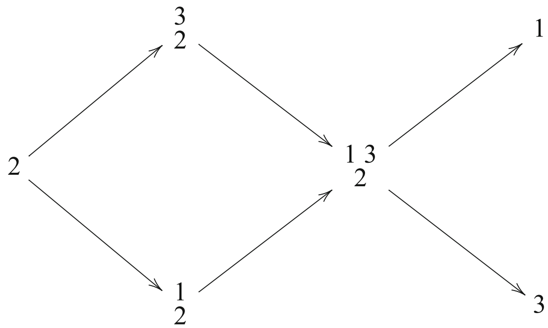
In this example, there are three almost split sequences:

$$0 \longrightarrow 2 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\ 3\\2\end{smallmatrix} \longrightarrow 0$$

$$0 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\ 3\\2\end{smallmatrix} \longrightarrow 3 \longrightarrow 0$$

$$0 \longrightarrow \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\ 3\\2\end{smallmatrix} \longrightarrow 1 \longrightarrow 0$$

and the Auslander–Reiten quiver is of the form



Let us point out that there are two further non-split short exact sequences with indecomposable end terms:

$$0 \longrightarrow 2 \longrightarrow \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow 3 \longrightarrow 0$$

$$0 \longrightarrow 2 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow 1 \longrightarrow 0$$

each of which can be obtained by "gluing the meshes" of two almost split sequences in the Auslander–Reiten quiver.

Problems

Exercises for Chap. 1

1.1. Let $M, M' \in \operatorname{rep} Q$. Show that the set of morphism $\operatorname{Hom}(M, M')$ is a k-vector space.

1.2. Let Q be the quiver $1 \longrightarrow 2 \longleftarrow 3$, and consider the representations

$$M \qquad k \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix}0\\1\end{bmatrix}} k$$

$$M' \qquad k \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} k$$

1. Show that M and M' are not indecomposable.
2. Show that M and M' are not isomorphic.

1.3. Let Q be the quiver $1 \longrightarrow 2 \longleftarrow 3$, and let M be the representation

$$k^2 \xrightarrow{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}} k^3 \xleftarrow{\begin{bmatrix}1\\0\\0\end{bmatrix}} k\ .$$

1. Write M as a direct sum of the indecomposable representations listed in Example 1.14.
2. Show that there is a non-split short exact sequence

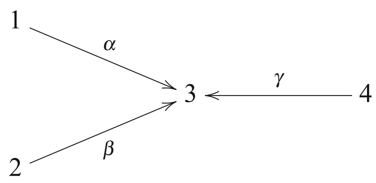
$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

such that $X \oplus Z = M$.

1.4. Find all indecomposable representations up to isomorphism of the quiver $1 \longrightarrow 2$. [Hint: Use the following theorem from Linear Algebra:]

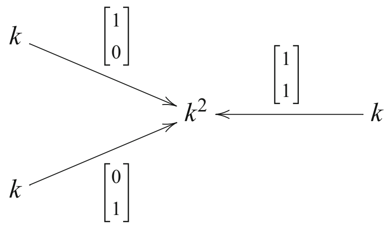
Theorem 1.16. *Let $\phi : V_1 \to V_2$ be a linear map between finite-dimensional vector spaces and fix some bases for V_1 and V_2. Let r be the rank of ϕ. Then there exist isomorphisms of vector spaces $f_i : V_i \to V_i$ such that the matrix of $f_2 \circ \phi \circ f_1^{-1}$ with respect to the fixed bases is a diagonal matrix whose upper left $r \times r$ block is the identity matrix and all other entries are zero.*

1.5. Let Q be the quiver

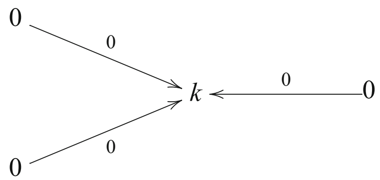


and consider the following representations:

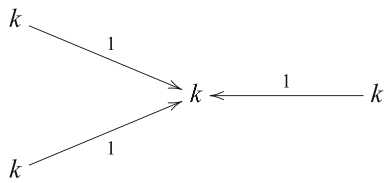
M



M'



M''



1. Show that

$$\mathrm{Hom}(M, M') = 0 \quad \mathrm{Hom}(M', M) \cong k^2$$
$$\mathrm{Hom}(M, M'') \cong k^2 \quad \mathrm{Hom}(M'', M) = 0.$$

2. Show that M is not isomorphic to $M' \oplus M''$.
3. Show that there is a short exact sequence:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

1.6. Let Q be the quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2\ .$$

For any $\lambda \in k \cup \{\infty\}$, define M_λ to be the representation:

$$k \underset{\lambda}{\overset{1}{\leftleftarrows}} k\ ;\ \text{if } \lambda \in k;$$

$$k \underset{1}{\overset{0}{\leftleftarrows}} k\ ;\ \text{if } \lambda = \infty.$$

1. Show that each M_λ is indecomposable.
2. Show that $M_\lambda \cong M_\mu$ if and only if $\lambda = \mu$. In particular, the number of indecomposable representations depends on the choice of the field k.
3. Show that $\mathrm{Hom}(M_\lambda, M_\mu) = 0$ if $\lambda \neq \mu$.
4. Show that for each λ there is a short exact sequence

$$0 \longrightarrow 1 \longrightarrow M_\lambda \longrightarrow 2 \longrightarrow 0,$$

where 1 and 2 are the representations

$$1:\ k \underset{0}{\overset{0}{\leftleftarrows}} 0 \quad , \quad 2:\ 0 \underset{0}{\overset{0}{\leftleftarrows}} k\ .$$

1.7. Let $f : L \to M$ be a morphism of representations.

1. Show that the cokernel of f together with the projection $\pi : M \to \operatorname{coker} f$ satisfies the universal property of Remark 1.6.
2. Show that if $g : M \to N$ is a morphism satisfying the universal property of Remark 1.6, then $N \cong \operatorname{coker} f$.

1.8. Let M, M', N be representations of Q and let $f : M \to N, g : M' \to N$ be morphisms. Define the *fiber product* (or *pull back*) of f and g as

$$X = \{(a, b) \mid a \in M, b \in M',\ \text{such that } f(a) = g(b)\},$$

and define the projections $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.

1. Show that X is a subrepresentation of $M \oplus M'$ and that there is a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi_2} & M' \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array}$$

2. Show that if f is injective then π_2 is injective.
3. Show that if f is surjective then π_2 is surjective.
4. Now suppose $0 \longrightarrow L \xrightarrow{h} M \xrightarrow{f} N \longrightarrow 0$ is a short exact sequence and define $h' : L \to X$ by $h'(n) = (h(n), 0)$. Show that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{h'} & X & \xrightarrow{\pi_2} & M' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_L} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & L & \xrightarrow{h} & M & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

1.9. Let L, M, M' be representations of Q and let $f : L \to M, g : L \to M'$ be morphisms. Define the *amalgamated sum* (or *push out*) of f and g as

$$X = (M \oplus M')/\{(f(\ell), -g(\ell)) \mid \ell \in L\},$$

and define the morphisms $u_1 \colon M \to X$ and $u_2 \colon M' \to X$ by $u_1(m) = \overline{(m, 0)}$ and $u_2(m') = \overline{(0, m')}$, where $\overline{(a, a')}$ denotes the class of $(a, a') \in M \oplus M'$ in X.

1. Show that there is a commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u_1} \\ M' & \xrightarrow{u_1} & X \end{array}$$

2. Now suppose $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{h} N \longrightarrow 0$ is a short exact sequence and define $h' : X \to N$ by $h'\overline{(m, m')} = h(m)$. Show that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{h} & N & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle u_1} & & \Big\downarrow{\scriptstyle 1_N} & & \\
0 & \longrightarrow & M' & \xrightarrow{u_2} & X & \xrightarrow{h'} & N & \longrightarrow & 0
\end{array}$$

1.10. Write out the morphisms of the 5 short exact sequences in Example 1.14. Show that the almost split sequence

$$0 \longrightarrow 2 \xrightarrow{h'} \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \xrightarrow{\pi_2} \begin{smallmatrix}1\ 3\\2\end{smallmatrix} \longrightarrow 0$$

is obtained from the short exact sequence

$$0 \longrightarrow 2 \xrightarrow{h} \begin{smallmatrix}1\\2\end{smallmatrix} \xrightarrow{f} 1 \longrightarrow 0$$

as in Exercise 1.8 via the fiber product of f and the irreducible morphism $g : \begin{smallmatrix}1\ 3\\2\end{smallmatrix} \to 1$.

Chapter 3
Examples of Auslander–Reiten Quivers

We have already pointed out in Sect. 1.5 that Auslander–Reiten quivers provide a threefold information about the representation theory of the quiver, namely the indecomposable representations, the irreducible morphisms, and the almost split sequences—these in turn should be thought of the building blocks of arbitrary representations, morphisms, and short exact sequences, respectively.

We have developed enough of the theory by now to be able to compute and appreciate Auslander–Reiten quivers. We present here several different methods of computation, although we are not able yet to prove that these methods actually produce the desired result; this justification is postponed to Chap. 7.

This chapter is subdivided into several sections. In the first section, we compute Auslander–Reiten quivers of type $\mathbb{A}_n$, the second section is a digression on finite representation type, and the third section treats the Auslander–Reiten quivers of type $\mathbb{D}_n$. In both the first and the third section, we present several methods to compute the Auslander–Reiten quiver.

The first method, the *knitting algorithm*, is a recursive procedure which owes its name to the fact that it produces one mesh after the other. The second method is to compute the orbits under the Auslander–Reiten translation τ. While the knitting algorithm produces the Auslander–Reiten quiver by computing the next vertical cross section and gradually progressing from left to right, the τ-orbit procedure computes horizontal cross sections of the Auslander–Reiten quiver. The third method is a geometric construction of the Auslander–Reiten quiver in terms of diagonals in a polygon in type $\mathbb{A}_n$ and in terms of arcs in a punctured polygon in type $\mathbb{D}_n$. We then show how to use the Auslander–Reiten quiver to compute the dimensions of Hom and Ext spaces between modules. In the fourth section, we introduce bound quivers and their representations in order to show how the geometric constructions for type $\mathbb{A}_n$ and $\mathbb{D}_n$ naturally generalize to the so-called cluster-tilted bound quivers. The reader who is not enthusiastic about the geometric realizations may very well skip the subsection on the punctured polygon (Sect. 3.3.3).

R. Schiffler, *Quiver Representations*, CMS Books in Mathematics,
DOI 10.1007/978-3-319-09204-1_3

3.1 Auslander–Reiten Quivers of Type $\mathbb{A}_n$

In this section, let Q be a quiver of type $\mathbb{A}_n$, that is, the underlying unoriented graph of Q is the Dynkin diagram of type $\mathbb{A}_n$:

$$1 \text{———} 2 \text{———} 3 \text{———} \dots \text{———} (n-1) \text{———} n.$$

We will see several ways to construct the Auslander–Reiten quiver of Q.

3.1.1 The Knitting Algorithm

The knitting algorithm owes its name to the fact that it recursively constructs one mesh after the other, from left to right. In order to get started one has to compute the indecomposable projective representations which are the leftmost indecomposable representations in the Auslander–Reiten quiver.

1. Compute the indecomposable projective representations

$$P(1), P(2), \dots, P(n).$$

2. Draw an arrow $P(i) \to P(j)$ whenever there exists an arrow $j \to i$ in Q_1, in such a way that each $P(i)$ sits at a different level.
3. (Knitting) There are three types of meshes. Complete each mesh as shown in Fig. 3.1 in such a way that

$$\underline{\dim}\, L + \underline{\dim}\, \tau^{-1} L = \sum_{i=1}^{2} \underline{\dim}\, M_i\,.$$

4. Repeat step 3 until you get negative integers in the dimension vector.

Observe that, every time we perform the third step, the representations L and M_i have been computed earlier and only $\tau^{-1}L$ is unknown.

The isoclasses of indecomposable representations of quivers of type $\mathbb{A}_n$ are determined by their dimension vectors as follows. The dimension vector is always of the form $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$, and the corresponding representation is $M = (M_i, \varphi_\alpha)$ with $M_i = k$ if the dimension at i is one, and $M_i = 0$ otherwise; and $\varphi_\alpha = 1$ if the dimension at $s(\alpha)$ and at $t(\alpha)$ is one, and $\varphi_\alpha = 0$ otherwise.

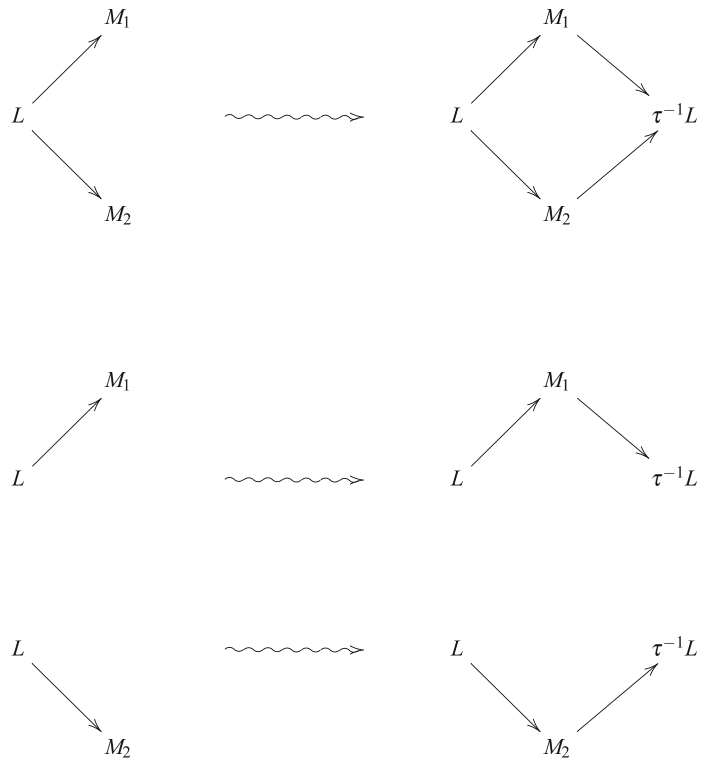


Fig. 3.1 Three types of meshes in the Auslander–Reiten quiver of type $\mathbb{A}_n$

Example 3.1. Let Q be the quiver

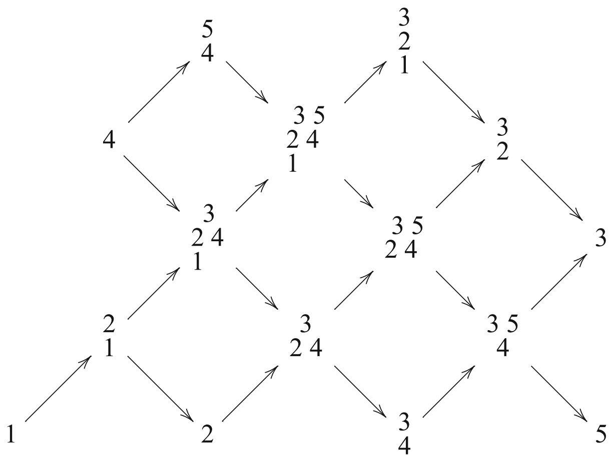
$$1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5\ .$$

Then

$$P(1) = 1 \quad P(2) = \begin{array}{c} 2 \\ 1 \end{array} \quad P(3) = \begin{array}{ccc} & 3 & \\ 2 & & 4 \\ 1 & & \end{array}$$

$$P(4) = 4 \quad P(5) = \begin{array}{c} 5 \\ 4 \end{array}$$

and the Auslander–Reiten quiver is



3.1.2 τ-Orbits

The map τ is the Auslander–Reiten translation. In the Auslander–Reiten quiver, it is the translation that sends the rightmost point of a mesh to the leftmost point of the same mesh. The τ-orbit of an indecomposable representation is the set of all representations that can be obtained by applying τ or τ^{-1} repeatedly to the representation. Thus the τ-orbits in the Auslander–Reiten quivers of type $\mathbb{A}_n$ consist of the representations that sit on the same level in the quiver.

Each τ-orbit in the Auslander–Reiten quiver of type $\mathbb{A}_n$ contains exactly one projective representation, so starting from the projectives, we can compute the whole quiver by computing the τ-orbits.

There are several methods to compute τ-orbits.

3.1.2.1 First Method: Auslander–Reiten Translation

Let M be an indecomposable representation that is not injective. We want to compute the translation to the right $\tau^{-1}M$ of M. Start with an injective resolution

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{g} I_1 \longrightarrow 0,$$

and apply the inverse Nakayama functor ν^{-1}. This functor maps the indecomposable injective representation $I(j)$ to the corresponding indecomposable projective representation $P(j)$; see Proposition 2.29 of Chap. 2. Then $\tau^{-1}M$ is given by the projective resolution:

$$0 \longrightarrow \nu^{-1}I_0 \xrightarrow{\nu^{-1}(g)} \nu^{-1}I_1 \longrightarrow \tau^{-1}M \longrightarrow 0.$$

Let us compute $\tau^{-1}M$ for the module $M = 4$ in Example 3.1. The upper line in the following diagram shows an injective resolution of M, and the lower line shows the corresponding projective resolution of $\tau^{-1}M$ obtained by applying ν^{-1}:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix}3\ 5\\4\end{smallmatrix} & \longrightarrow & 3\oplus 5 & \longrightarrow & 0 & & \\ & & & & \downarrow{\scriptstyle \nu^{-1}} & & \downarrow{\scriptstyle \nu^{-1}} & & & & \\ & & 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix}3\\2\ 4\\1\end{smallmatrix}\oplus\begin{smallmatrix}5\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\ 5\\2\ 4\\1\end{smallmatrix} & \longrightarrow & 0 \end{array}$$

Thus $\tau^{-1}M = \begin{smallmatrix}3\ 5\\2\ 4\\1\end{smallmatrix}$ which verifies the result of Example 3.1.

3.1.2.2 Second Method: Coxeter Functor

Choose a sequence of vertices $(i_1, i_2, \ldots, i_n)$, with $i_j \neq i_\ell$ if $i \neq \ell$, as follows:

i_1 is a sink of Q;
i_2 is a sink of the quiver $s_{i_1}Q$ obtained from Q by reversing all arrows that are incident to the vertex i_1;
i_t is a sink of $s_{i_{t-1}} \ldots s_{i_2}s_{i_1}Q$, for $t = 2, 3, \ldots, n$.

Thus in Example 3.1 such a sequence would be $(1, 4, 2, 3, 5)$.

Next, we need the notion of reflections $s_i : \mathbb{R}^n \to \mathbb{R}^n$ defined by $s_i(x) = x - 2B(x, e_i)e_i$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ and B is a symmetric bilinear form defined by

$$B(e_i, e_j) = \begin{cases} 1 & \text{if } i = j \\ -1/2 & \text{if } i \text{ is adjacent to } j \text{ in } Q \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $s_i(\sum_j a_j e_j) = \sum_j a'_j e_j$, where $a'_j = a_j$ if $j \neq i$ and $a'_i = -a_i + \sum_{i \,—\, j} a_j$, where the sum is over all vertices j that are adjacent to i in Q.

Finally, we define a so-called *Coxeter element* $c = s_{i_1}s_{i_2}\cdots s_{i_n}$ as a product of reflections using the sequence of vertices defined above. Thus in Example 3.1 such a Coxeter element would be $c = s_1s_4s_2s_3s_5$.

One can use this Coxeter element to compute the dimension vector of the representation $\tau^{-1}M$ from the dimension vector of M. If $\underline{\dim}M = (d_1, d_2, \ldots, d_n)$, then $c(\sum_i d_i e_i) = \sum_i d_i' e_i$ and $\underline{\dim}(\tau^{-1}M) = (d_1', d_2', \ldots, d_n')$.

Let us use this method to compute the dimension vector of $\tau^{-1}4$ in Example 3.1. We have $\underline{\dim}\, M = (0, 0, 0, 1, 0)$. Thus $\underline{\dim}\, \tau^{-1}M$ is equal to

$$\begin{aligned} s_1 s_4 s_2 s_3 s_5 (e_4) &= s_1 s_4 s_2 s_3 (e_4 + e_5) \\ &= s_1 s_4 s_2 (e_3 + e_4 + e_5) \\ &= s_1 s_4 (e_2 + e_3 + e_4 + e_5) \\ &= s_1 (e_2 + e_3 + e_4 + e_5) \\ &= e_1 + e_2 + e_3 + e_4 + e_5, \end{aligned}$$

which again confirms the result obtained in Example 3.1.

Another way of defining the action of the Coxeter element is to use the **Cartan matrix** C of the quiver Q. This matrix is defined as $C = (c_{ij})_{1\le i,j \le n}$, where c_{ij} is the number of paths from j to i and n is the number of vertices in Q. It follows directly from the definition that, for every vertex i, the ith column of C is exactly the dimension vector of the indecomposable projective representation $P(i)$ and the ith row of C is exactly the dimension vector of the indecomposable injective representation $I(i)$.

Since Q has no oriented cycles, we can always renumber the vertices of Q in such a way that, if there is a path from j to i, then $i \le j$; in other words, there is a renumbering of the vertices such that the matrix C is upper triangular. Also note that the diagonal entries of C are all equal to 1, since there is exactly one path, the constant path, from each vertex to itself. This shows that C is invertible.

Its inverse C^{-1} is the matrix $(b_{ij})_{1\le i,j\le n}$ where $b_{ii} = 1$, and if $i \ne j$, then $-b_{ij}$ is the number of arrows from j to i in Q. To show that this is indeed the inverse of C, we multiply the two matrices:

$$(c_{ij})_{i,j}(b_{j\ell})_{j,\ell} = \left(\sum_j c_{ij} b_{j\ell} \right)_{i,\ell} .$$

Note first that the diagonal entries $\sum_j c_{ij} b_{ji} = c_{ii} b_{ii} = 1$, since both matrices are upper triangular (up to some renumbering of the vertices). Next, if $i \ne \ell$, then each path from ℓ to i must start with some arrow from ℓ to some vertex j. Therefore, the number $c_{i\ell}$ of paths from ℓ to i can be computed as

$$c_{i\ell} = \sum_{j \in Q_0 \setminus \{\ell\}} c_{ij}(-b_{j\ell}).$$

Now using $b_{\ell\ell} = 1$, we have $\sum_j c_{ij} b_{j\ell} = c_{i\ell} + \sum_{j\ne\ell} c_{ij} b_{j\ell} = 0$ if $i \ne \ell$. Thus $C^{-1} = (b_{ij})_{1\le i,j\le n}$.

Now we define yet another matrix, the **Coxeter matrix** Φ, as $\Phi = -C^{\top}(C^{-1})$, and its inverse is $\Phi^{-1} = -C(C^{-1})^{\top}$, the superscript $^{\top}$ here denotes the transpose of a matrix. Then

$$\Phi\, \underline{\dim}\, M = \underline{\dim}\, \tau M, \text{ if } M \text{ is not projective and } \Phi\, \underline{\dim}\, P(j) = -\underline{\dim}\, I(j),$$

whereas

$$\Phi^{-1}\underline{\dim}\, M = \underline{\dim}\, \tau^{-1} M, \text{ if } M \text{ is not injective and } \Phi^{-1}\underline{\dim}\, I(j) = -\underline{\dim}\, P(j).$$

In our Example 3.1, we have

$$C = \begin{bmatrix} 1&1&1&0&0 \\ 0&1&1&0&0 \\ 0&0&1&0&0 \\ 0&0&1&1&1 \\ 0&0&0&0&1 \end{bmatrix} \quad (C^{-1}) = \begin{bmatrix} 1&-1&0&0&0 \\ 0&1&-1&0&0 \\ 0&0&1&0&0 \\ 0&0&-1&1&-1 \\ 0&0&0&0&1 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} -1&1&0&0&0 \\ -1&0&1&0&0 \\ -1&0&1&-1&1 \\ 0&0&1&-1&1 \\ 0&0&1&-1&0 \end{bmatrix} \quad \Phi^{-1} = \begin{bmatrix} 0&0&-1&1&0 \\ 1&0&-1&1&0 \\ 0&1&-1&1&0 \\ 0&1&-1&1&-1 \\ 0&0&0&1&-1 \end{bmatrix}$$

so that the dimension of $\tau^{-1}4$ can be computed by $\Phi^{-1}\,(0,0,0,1,0)^{\top}$ which is equal to $(1,1,1,1,1)^{\top}$. On the other hand, $\Phi\, \underline{\dim}\, P(4) = \Phi\,(0,0,0,1,0)^{\top} = (0,0,-1,-1,-1)^{\top} = -\underline{\dim}\, I(4)$.

3.1.3 Diagonals of a Polygon with n + 3 Vertices

In this section, we give a geometric way to construct the Auslander–Reiten quiver of a quiver Q of type $\mathbb{A}_n$ from a triangulation of a polygon. This method works only for quivers of type $\mathbb{A}_n$.

Start with a regular polygon with $n + 3$ vertices. A *diagonal* in the polygon is a straight line segment that joins two of the vertices and goes through the interior of the polygon, and a *triangulation* of the polygon is a maximal set of non-crossing diagonals. Such a triangulation cuts the polygon into triangles, hence the name. Given a triangle with sides a, b, c, we say that the side a is clockwise of the side b if going along the boundary of the triangle in the clockwise direction corresponds to the sequence $a, b, c, a, b, c, a \dots$.

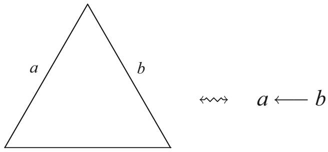


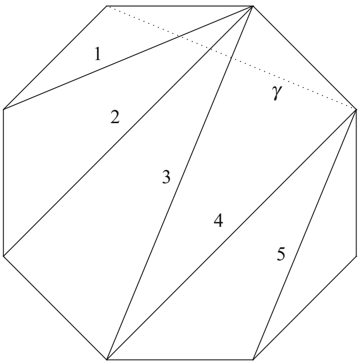
Fig. 3.2 b is clockwise from a corresponding to an arrow from b to a

We will associate a triangulation T_Q to our type $\mathbb{A}_n$ quiver Q as follows: Let 1 be a vertex in the quiver that has only one neighbor. Draw a diagonal that cuts off a triangle Δ_0 and label that diagonal 1. If $1 \leftarrow 2$ is an arrow in Q, then draw the unique diagonal 2 such that 1, 2 and one boundary segment of the polygon form a triangle Δ_1 in such a way that diagonal 2 is clockwise of diagonal 1 in the triangle Δ_1. If, on the other hand, $1 \rightarrow 2$ is an arrow in Q, draw the unique diagonal 2 such that diagonal 2 is counterclockwise of diagonal 1 in the triangle Δ_1; see Fig. 3.2. Continue this procedure up to diagonal n.

In this way the quiver

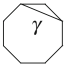
$$1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$$

of Example 3.1 gives rise to the triangulation



Since T_Q is a triangulation of the polygon, any other diagonal γ which is not already in T_Q will cut through a certain number of diagonals in T_Q; in fact, any such diagonal γ is uniquely determined by the set of diagonals in T_Q that γ crosses. To such a diagonal γ, we associate a representation $M_\gamma = (M_i, \varphi_\alpha)$ of Q by letting

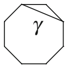
$$M_i = \begin{cases} \mathrm{k} \text{ if the diagonal } \gamma \text{ crosses the diagonal } i; \\ 0 \text{ otherwise;} \end{cases}$$

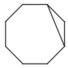
and setting $\varphi_\alpha = 1$ whenever $M_{s(\alpha)} = M_{t(\alpha)} = k$, and $\varphi_\alpha = 0$ otherwise. In the example, the diagonal 

crosses the diagonals 1, 2, and 3, and the corresponding representation is

$$k \xleftarrow{\;1\;} k \xleftarrow{\;1\;} k \xrightarrow{\;0\;} 0 \xleftarrow{\;0\;} 0\ .$$

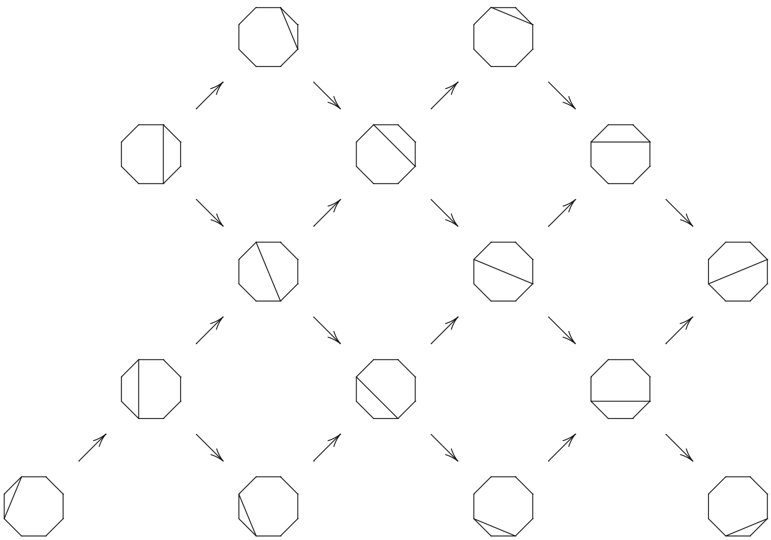
The map $\gamma \mapsto M_\gamma$ is a bijection from the set of diagonals that are not in T_Q and the set of isoclasses of indecomposable representations of Q.

The Auslander–Reiten translation τ is given by an elementary clockwise rotation of the polygon, so in our example τ of 

is the diagonal  that cuts through the diagonals 4 and 5.

The projective representation $P(i)$ is given by τ^{-1} of the diagonal i, and the injective representation $I(i)$ is given by τ of the diagonal i. In our example $P(1)$ is the diagonal that cuts through the diagonal 1 only and $I(1)$ is the diagonal γ.

The complete Auslander–Reiten quiver can be easily constructed now starting with the projectives and applying the elementary rotation to compute the τ-orbits until we reach the injective in each τ-orbit, and the Auslander–Reiten quiver is



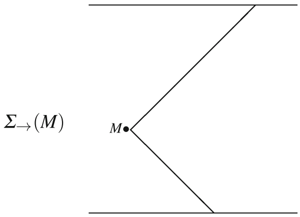
Note that any arrow in the Auslander–Reiten quiver acts on the diagonal by pivoting one of the endpoints of the diagonal to its counterclockwise neighbor.

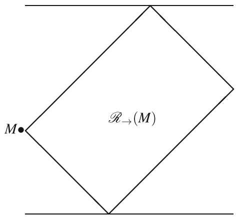
3.1.4 Computing Hom Dimensions, Ext Dimensions, and Short Exact Sequences

Given two indecomposable representations M and N, we want to have information about the space of morphisms $\mathrm{Hom}(M, N)$. The Auslander–Reiten quiver allows us to compute the dimension of this space easily, at least if M and N lie in the same connected component.

3.1.4.1 Dimension of Hom(M, N)

Let Q be a type $\mathbb{A}$ quiver and let M, N be two indecomposable representations of Q. We can compute the dimension of the vector space $\mathrm{Hom}(M, N)$ using the relative position of M and N in the Auslander–Reiten quiver. For this we need to introduce some terminology:





A path $M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_s$ in the Auslander–Reiten quiver is called a **sectional path** if $\tau M_{i+1} \neq M_{i-1}$ for all $i = 1, \ldots, s-1$. Let $\Sigma_{\rightarrow}(M)$ be the set of all indecomposable representations that can be reached from M by a sectional path, and let $\Sigma_{\leftarrow}(M)$ be the set of all indecomposable representations from which one can reach M by a sectional path.

Now let $\mathscr{R}_{\rightarrow}(M)$ be the set of all indecomposable representations whose position in the Auslander–Reiten quiver is in the slanted rectangular region whose left boundary is $\Sigma_{\rightarrow}(M)$. We call $\mathscr{R}_{\rightarrow}(M)$ the maximal slanted rectangle in the Auslander–Reiten quiver whose leftmost point is M. Then $\dim \mathrm{Hom}(M, N)$ is either 1 or 0, and it is 1 if and only if N lies in $\mathscr{R}_{\rightarrow}(M)$.

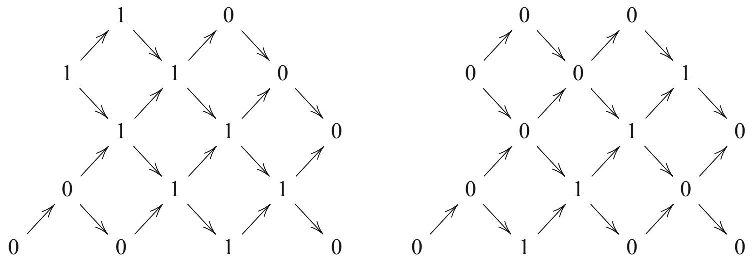


Fig. 3.3 Dimension of $\mathrm{Hom}(M, -)$ for $M = P(4)$ on the *left* and $M = S(2)$ on the *right*. The position of the representation M is at the leftmost 1 in each case; the numbers 0, 1 indicate the dimension of $\mathrm{Hom}(M, N)$ for each indecomposable representation N

We illustrate this concept in Fig. 3.3 for the Auslander–Reiten quiver of Example 3.1. On the left side of Fig. 3.3, the module M is the indecomposable projective $P(4)$. Its position in the Auslander–Reiten quiver is the leftmost 1 in the figure, so this 1 indicates that $\dim \mathrm{Hom}(M, M) = 1$. A basis for this vector space is the identity morphism 1_M. Each indecomposable representation N is located at a specific point in the Auslander–Reiten quiver; the number 0 or 1 at that point indicates the dimension of $\mathrm{Hom}(M, N)$ for each N.

In the Auslander–Reiten quiver on the right-hand side of Fig. 3.3, the module M is the simple module $S(2)$. Again its position is the leftmost 1 in that figure. The rectangle on which $\mathrm{Hom}(M, -)$ is nonzero reduces in this case to a single line.

Symmetrically, we denote by $\mathscr{R}_{\leftarrow}(N)$ the maximal slanted rectangle in the Auslander–Reiten quiver whose rightmost point is N. We can compute the dimension of $\mathrm{Hom}(-, N)$ using $\mathscr{R}_{\leftarrow}(N)$. Thus the data in the left picture in Fig. 3.3 also computes the $\dim \mathrm{Hom}(-, N)$ for $N = I(4)$.

Note that if $M = P(i)$ is an indecomposable projective, then it follows from Theorem 2.11 that the representations in $\mathscr{R}_{\rightarrow}(P(i))$ are precisely the indecomposable representations N such that $N_i \neq 0$. It then follows from Exercise 2.7 of Chap. 2 that there is a unique rightmost point in $\mathscr{R}_{\rightarrow}(P(i))$ which must be the position of the indecomposable injective representation $I(i)$. In particular, $\mathscr{R}_{\rightarrow}(P(i)) = \mathscr{R}_{\leftarrow}(I(i))$.

Figure 3.4 shows an example where the right end of the $\mathscr{R}_{\rightarrow}(M)$ does not really have the shape of a rectangle, because the Auslander–Reiten quiver ends before the rectangle is completed. This happens exactly when M is not projective.

3.1.4.2 Dimension of $\mathbf{Ext}^1(M, N)$

Next we compute the dimensions of the vector spaces $\mathrm{Ext}^1(M, N)$ for indecomposable representations M, N of type $\mathbb{A}$. If M is projective, then this space is zero, by Exercise 2.11 of Chap. 2, so let us assume that M is not projective. Thus τM is a

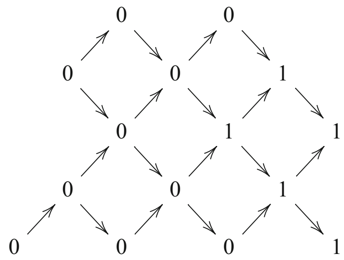


Fig. 3.4 Dimension of $\text{Hom}(M, -)$ where M is the representation whose dimension vector is $(0, 1, 1, 1, 1)$

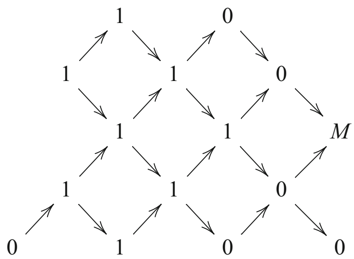


Fig. 3.5 Dimension of $\text{Ext}^1(M, -)$ for $M = I(3)$

point in the Auslander–Reiten quiver. We will see in Theorem 7.18 that there is an isomorphism

$$\text{Ext}^1(M, N) \cong D\text{Hom}(N, \tau M),$$

where D is the duality and τ is the Auslander–Reiten translation. This isomorphism implies that $\dim \text{Ext}^1(M, N) = \dim \text{Hom}(N, \tau M)$ and therefore we can compute the dimension of $\text{Ext}^1(M, -)$ using the maximal slanted rectangle $\mathscr{R}_{\leftarrow}(\tau M)$.

Figure 3.5 shows the dimension of $\text{Ext}^1(M, -)$ for the representation $M = I(3)$ in our running example.

3.1.4.3 Short Exact Sequences

We have seen in Sect. 2.4 of Chap. 2 that the elements of $\text{Ext}^1(M, N)$ can be represented by short exact sequences of the form $0 \to N \to E \to M \to 0$, where E is some representation of Q. We are interested here in the case where M and N are indecomposable—this does *not* imply that E is indecomposable.

We now want to compute the possible representations E for these short exact sequences. If the dimension of $\text{Ext}^1(M, N)$ is 0, then the only possibility is $E \cong M \oplus N$. If on the other hand, the dimension of $\text{Ext}^1(M, N)$ is 1, then, up to

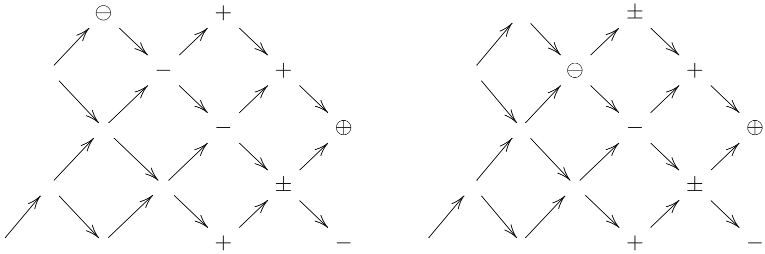
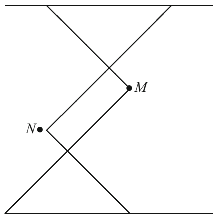


Fig. 3.6 Computing short exact sequences

isomorphism, there is exactly one other possibility for E. For representations of type $\mathbb{A}$, we can compute E simply from the relative positions of M and N in the Auslander–Reiten quiver:

$\Sigma_{\rightarrow}(N)$ *and* $\Sigma_{\leftarrow}(M)$



Let M, N be indecomposable representations of a quiver of type $\mathbb{A}$ such that $\operatorname{Ext}^1(M, N) \neq 0$. Then N must lie in $\mathscr{R}_{\leftarrow}(\tau M)$ and this implies that $\Sigma_{\rightarrow}(N)$ and $\Sigma_{\leftarrow}(M)$ have either 1 or 2 points in common, and these points correspond to the indecomposable summands of E.

We illustrate this situation in Fig. 3.6; the representation N is marked by $\ominus$ and the representation M by $\oplus$. The representations in $\Sigma_{\rightarrow}(N)$ are marked by $-$ or $\ominus$ (for N) and those in $\Sigma_{\leftarrow}(M)$ by $+$ or $\oplus$ (for M). The points of intersection are marked $\pm$. The example of the left-hand side of Fig. 3.6 corresponds to the short exact sequence:

$$0 \longrightarrow \begin{smallmatrix} 5 \\ 4 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 3 & & 5 \\ & 4 & \end{smallmatrix} \longrightarrow 3 \longrightarrow 0$$

and the example on the right-hand side corresponds to the short exact sequence:

$$0 \longrightarrow \begin{smallmatrix} & 3 & & 5 \\ 2 & & 4 & \\ 1 & & & \end{smallmatrix} \longrightarrow \begin{smallmatrix} 3 & & 5 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \longrightarrow 3 \longrightarrow 0\,.$$

3.2 Representation Type

3.2.1 Gabriel's Theorem: Finite Representation Type

A quiver Q is said to be of **finite representation type** if the number of isoclasses of indecomposable representations of Q is finite. In this section, we list the quivers of finite representation type. It turns out that this classification depends only on the shape of the quiver and not on the particular orientation of the arrows. We therefore define the **underlying graph** of the quiver Q to be the graph obtained from Q by forgetting the direction of the arrows; thus the underlying graph has the same vertices as Q and for each arrow $i \to j$ in Q there is an edge $i — j$ in the underlying graph.

The graphs in Fig. 3.7 are called **Dynkin diagrams**. These graphs play an important role in mathematics when it comes to classifications. There are four infinite series, types $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and $\mathbb{D}$, and five exceptional diagrams, types $\mathbb{E}, \mathbb{F}$ and $\mathbb{G}$. The types $\mathbb{A}, \mathbb{D}, \mathbb{E}$ are the only ones that have no parallel edges; these types are called **simply laced** Dynkin diagrams and will be of particular interest to us. The classification result is as follows:

Theorem 3.1 (Gabriel's Theorem, Part I). *A connected quiver is of finite representation type if and only if its underlying graph is one of the Dynkin diagrams of type* $\mathbb{A}, \mathbb{D}$ *or* $\mathbb{E}$.

This is a very surprising result, one might of course ask now what is so special about the Dynkin diagrams, or why are there only three diagrams of type $\mathbb{E}$? Note that we cannot come up with a diagram of $\mathbb{E}$ type with five or less vertices, because it would be a diagram of type $\mathbb{D}$ or $\mathbb{A}$. But what about $\mathbb{E}$ type diagrams with $9, 10$, or more vertices? Well, the simple answer is that you then get infinitely many indecomposable representations, but this answer does not really settle the question: why?

One thing we can say is that we are not the only ones who are puzzled about this fact, because the Dynkin diagrams show up in finite type classifications of objects in several different fields of mathematics, for example, in the classifications of Lie algebras, root systems, Coxeter groups, and cluster algebras. These diagrams just happen to be very fundamental objects that reflect finite type structures that arise in nature.

We postpone the proof of Gabriel's theorem to Chap. 8. For now we just want to use it to move beyond type $\mathbb{A}$ in our section on examples of Auslander–Reiten quivers. From Gabriel's theorem we see that we should compute the $\mathbb{D}$-type next. This is done in the following section.

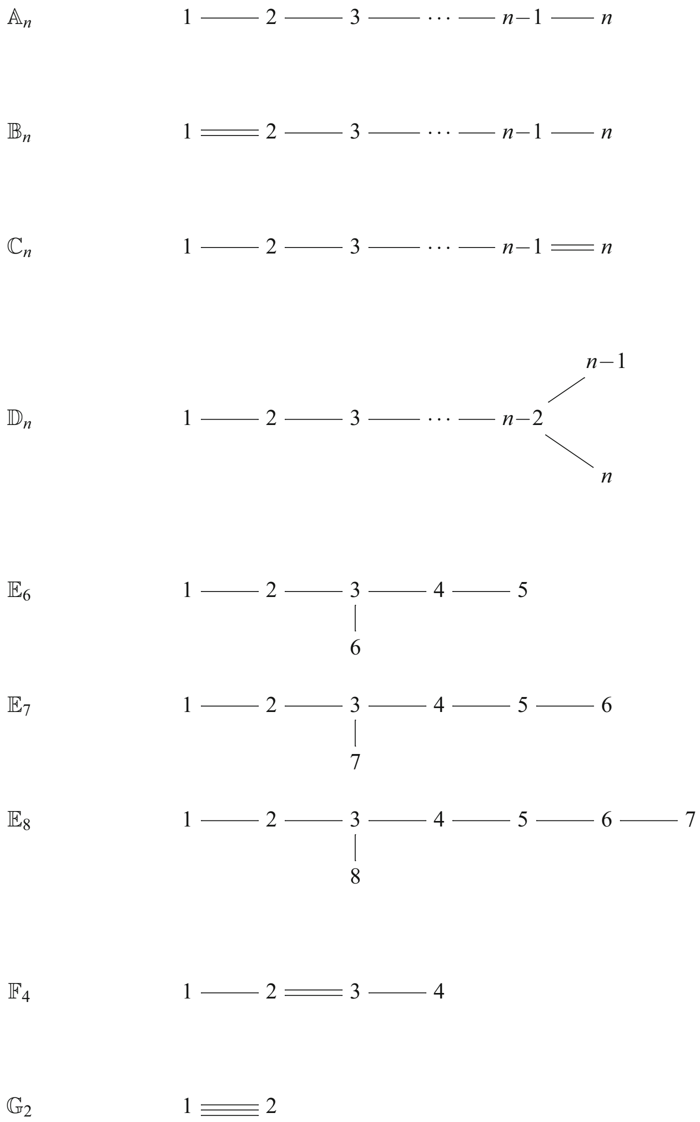


Fig. 3.7 Dynkin diagrams

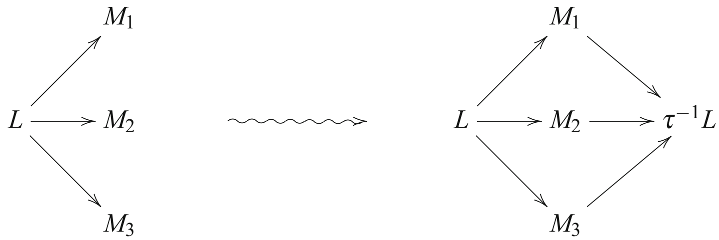
3.3 Auslander–Reiten Quivers of Type $\mathbb{D}_n$

In this section, let Q be a quiver of type $\mathbb{D}_n$, that is, the underlying unoriented graph of Q is the Dynkin diagram of type $\mathbb{D}_n$.

We will use the different techniques from Sect. 3.1 to construct the Auslander–Reiten quiver of Q.

3.3.1 The Knitting Algorithm

We can use this algorithm in almost the same way as for type $\mathbb{A}_n$, with the difference that now, there is a fourth type of mesh:



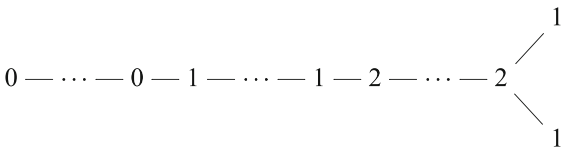
The isoclasses of indecomposable representations of quivers of type $\mathbb{D}_n$ are determined by their dimension vectors $\mathbf{d} = (d_1, \ldots, d_n)$ as follows. The entries d_i of the dimension vector are either 0, 1 or 2, and if we have $d_i = 2$, then

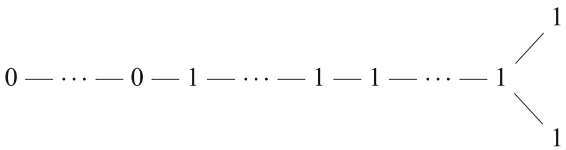
1. i is one of the vertices $2, 3, \ldots, n-2$,
2. for all vertices j with $i \leq j \leq n-2$ we have $d_j = 2$,
3. $d_{i-1} \geq 1$ and $d_{n-1} = d_n = 1$.

Thus the vertices i with $d_i = 2$ form a subgraph of type $\mathbb{A}$ that contains the vertex $n-2$.

The vertices $i \neq n-1, n$ with $d_i = 1$ also form a subgraph of type $\mathbb{A}$, and if $d_j \neq 2$ for all j, then all the vertices i with $d_i = 1$ form a subgraph of type $\mathbb{A}$ or a subgraph of type $\mathbb{D}$.

Graphically, we can represent some of these configurations as follows:





The corresponding representation is $M = (M_i, \varphi_\alpha)$ with $M_i = k^{d_i}$; and $\varphi_\alpha = 1$ if $d_{s(\alpha)} = d_{t(\alpha)}$, $\varphi_\alpha = 0$ if one of $d_{s(\alpha)}, d_{t(\alpha)}$ is zero. If one of the d_i is 2, then there are exactly three arrows that connect a vertex with dimension 1 to a vertex with dimension 2: two of these arrows, let us call them β_1, β_2, connect the vertex $n-2$ with the vertices $n-1$ and n, the vector space of dimension two being at $n-2$, while the third arrow α_i connects two vertices i and $i+1$, the vector space of dimension two being at vertex $i+1$. Consider the one-dimensional subspace of M_{i+1} given by

$$\begin{cases} \operatorname{im}\varphi_{\alpha_i} & \text{if } \alpha_i \text{ points to } i+1, \\ \ker\varphi_{\alpha_i} & \text{otherwise.} \end{cases}$$

Under the composition of the identity maps $\varphi_{\alpha_{n-3}} \cdots \varphi_{\alpha_{i+1}}$ this one-dimensional subspace is sent to a one-dimensional subspace ℓ_1 of M_{n-2}. Consider also the following two one-dimensional subspaces ℓ_2 and ℓ_3 of M_{n-2}:

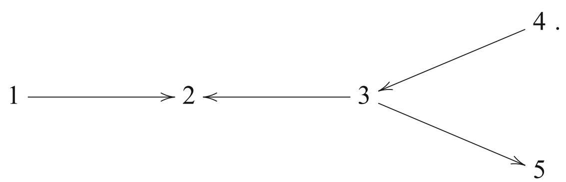
$$\ell_2 = \begin{cases} \operatorname{im}\varphi_{\beta_1} & \text{if } \beta_1 \text{ points to } n-2, \\ \ker\varphi_{\beta_1} & \text{otherwise;} \end{cases}$$

and

$$\ell_3 = \begin{cases} \operatorname{im}\varphi_{\beta_2} & \text{if } \beta_2 \text{ points to } n-2, \\ \ker\varphi_{\beta_2} & \text{otherwise.} \end{cases}$$

Then the condition on the three maps $\varphi_{\alpha_i}, \varphi_{\beta_1}$ and φ_{β_2} is that the three one-dimensional subspaces are pairwise distinct. This corresponds to the "generic" situation as opposed to the special case where two (or more) of these subspaces are equal.

Example 3.2. Let Q be the quiver

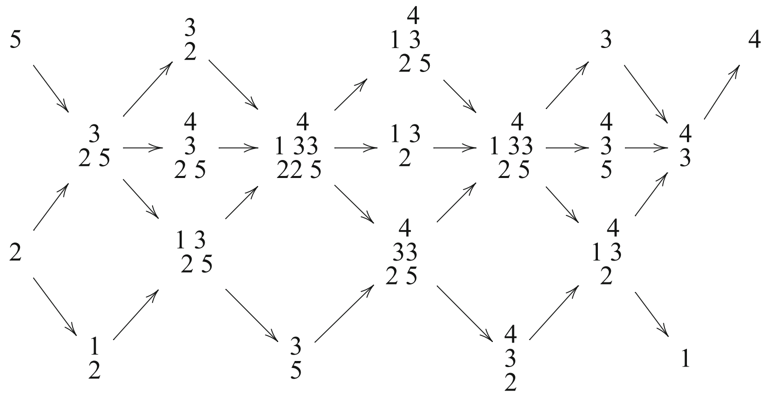


Then

$$P(1) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \quad P(2) = 2 \quad P(3) = \begin{smallmatrix} & 3 & \\ 2 & & 5 \end{smallmatrix}$$

$$P(4) = \begin{smallmatrix} & 4 & \\ & 3 & \\ 2 & & 5 \end{smallmatrix} \quad P(5) = 5$$

and, using the knitting algorithm, the Auslander–Reiten quiver is

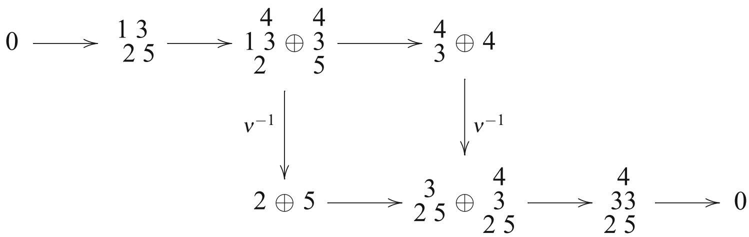


3.3.2 τ-Orbits

As in type A, there are several ways to compute the τ-orbits.

3.3.2.1 First Method: Auslander–Reiten Translation

Let us compute $\tau^{-1}M$ for the module $M = \begin{smallmatrix}1\,3\\ 2\,5\end{smallmatrix}$ in Example 3.2. The upper line in the following diagram shows an injective resolution of M, and the lower line shows the projective resolution of $\tau^{-1}M$ obtained by applying ν^{-1}.



Thus $\tau^{-1}M = \begin{smallmatrix}4\\ 33\\ 2\,5\end{smallmatrix}$ which verifies the result of Example 3.2.

3.3.2.2 Second Method: Coxeter Functor

As in Sect. 3.1.2.2, we define a sequence of vertices $(i_1, i_2, \ldots, i_n)$, with $i_j \neq i_\ell$, if $j \neq \ell$, as follows.

i_1 is a sink of Q.
i_2 is a sink of the quiver $s_{i_1} Q$ obtained from Q by reversing all arrows that are incident to the vertex i.
i_k is a sink of $s_{i_{k-1}} \ldots s_{i_2} s_{i_1} Q$, for $k = 2, 3, \ldots, n$.

Then we define the Coxeter element $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ as a product of reflections using this sequence of vertices.

Thus in Example 3.2, we can take the sequence $(2, 5, 1, 3, 4)$, and its Coxeter element is $c = s_2 s_5 s_1 s_3 s_4$.

Let us use this Coxeter element to compute the dimension vector of $\tau^{-1} \begin{smallmatrix} 1\,3 \\ 2\,5 \end{smallmatrix}$ in Example 3.2. We have $\underline{\dim}\, M = (1, 1, 1, 0, 1)$. Thus $\underline{\dim}\, \tau^{-1} M$ is equal to

$$\begin{aligned} s_2 s_5 s_1 s_3 s_4 (e_1 + e_2 + e_3 + e_5) &= s_2 s_5 s_1 s_3 (e_1 + e_2 + e_3 + e_4 + e_5) \\ &= s_2 s_5 s_1 (e_1 + e_2 + 2e_3 + e_4 + e_5) \\ &= s_2 s_5 (e_2 + 2e_3 + e_4 + e_5) \\ &= s_2 (e_2 + 2e_3 + e_4 + e_5) \\ &= e_2 + 2e_3 + e_4 + e_5 \end{aligned}$$

which again confirms the result obtained in Example 3.2.

As in type $\mathbb{A}$, we can also use the Cartan matrix C and the Coxeter matrix $\Phi = -C^t C^{-1}$ in order to compute the action of the Coxeter element. In our example, we have

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad (C^{-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \quad \Phi^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

Thus for the representation M above, we can compute the dimension vector of $\tau^{-1} M$ as $\Phi^{-1} \underline{\dim}\, M = \Phi^{-1}(1, 1, 1, 0, 1)^t = (0, 1, 2, 1, 1)^t$.

On the other hand, τM has dimension vector $\Phi(1, 1, 1, 0, 1)^t = (0, 1, 0, 0, 0)^t$.

3.3.3 Arcs of a Punctured Polygon with n Vertices

In this section, we give a geometric construction of the Auslander–Reiten quiver of a quiver Q of type D_n similar to the construction in Sect. 3.1.3. Instead of a triangulated polygon, we work with a triangulated punctured polygon. Thee diagonals in the polygon must be replaced by certain curves that are called *arcs* in the puncture polygon. If the boundary of the polygon has n vertices, then we have exactly n^2 arcs given as follows:

For every vertex a on the boundary of the polygon, we have the $n-2$ arcs shown in the left picture of Fig. 3.8, and for the puncture, we have the n arcs shown in the middle and the n arcs shown in the right picture of Fig. 3.8. Note that for each boundary vertex a, there are two arcs from a to the puncture, and we use a little tag on the arc to distinguish them. The arcs at the puncture that have a tag are called *notched* and the ones without a tag are called *plain*.

Also note that, given two boundary vertices $a \neq b$, there is exactly one arc connecting a and b if a and b are neighbors on the boundary and exactly two arcs if a and b are not neighbors, see Fig. 3.9.

Contrary to the case of the diagonals in the polygon, it is not so straightforward to say when two arcs γ and γ' in the punctured polygon cross.

We denote the number of crossings by $e(\gamma, \gamma')$. If one of the two arcs has both endpoints on the boundary of the polygon, the number of crossing should be

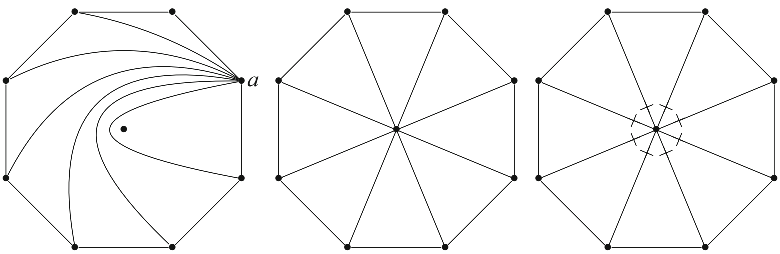


Fig. 3.8 Arcs in a punctured polygon with eight boundary vertices

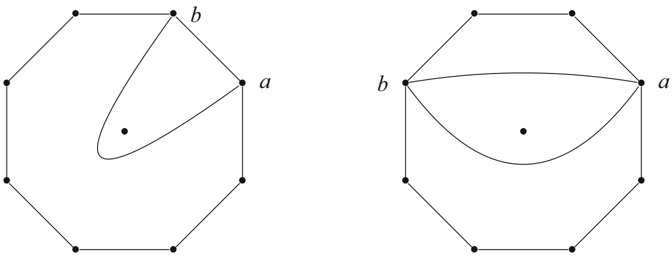


Fig. 3.9 Arcs with specified endpoints

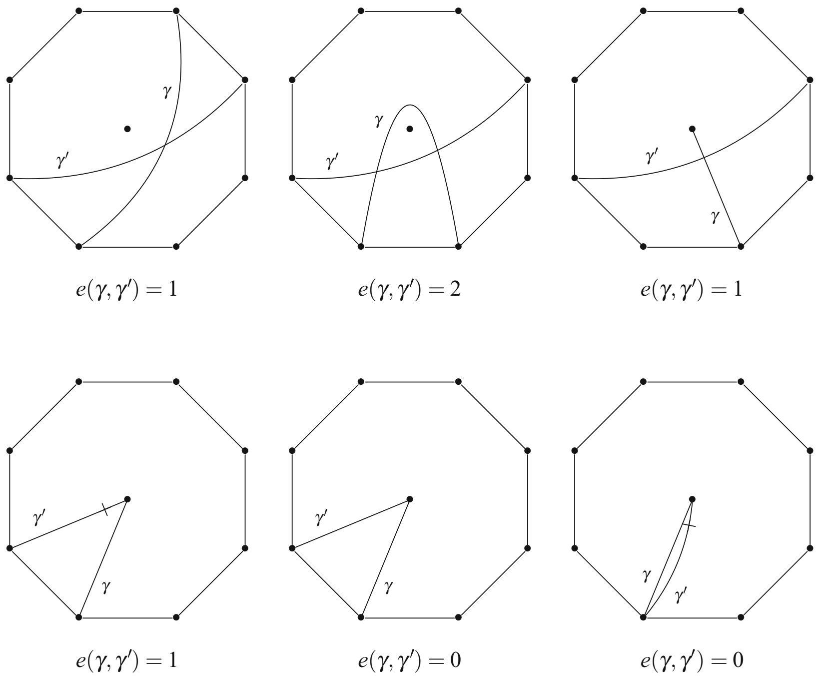


Fig. 3.10 Crossing numbers

intuitively clear, and we show several examples in Fig. 3.10. Note that in this case $e(\gamma, \gamma')$ can be 0, 1, or 2. For a rigorous definition of crossing numbers we would need the notion of homotopy, which would take us too far away from the subject of this book.

If both arcs γ and γ' are incident to the puncture and a and a' denote their respective endpoints on the boundary, we define

$$e(\gamma, \gamma') = \begin{cases} 0 \text{ if } \gamma \text{ and } \gamma' \text{ are both plain,} \\ 0 \text{ if } \gamma \text{ and } \gamma' \text{ are both notched,} \\ 0 \text{ if } a = a', \\ 1 \text{ if } \gamma, \gamma' \text{ have opposite tagging and } a \neq a'. \end{cases}$$

We say that two arcs cross if their crossing number is at least 1, and a *triangulation* is a maximal set of non-crossing diagonals. A triangulation does not necessarily cut the polygon into triangles, even if one allows triangles to have curved edges. Some triangulations are shown in Fig. 3.11.

Now let Q be a quiver of Dynkin type $\mathbb{D}_n$. We associate a triangulation T_Q to Q as follows: Start with an arc γ_1 that cuts off a triangle Δ_0. If $1 \leftarrow 2$ is in Q, then let γ_2 be the unique arc that forms a triangle Δ_1 together with γ_1 and a boundary segment in such a way that γ_1 is counterclockwise from γ_2 in Δ_1. If on the other

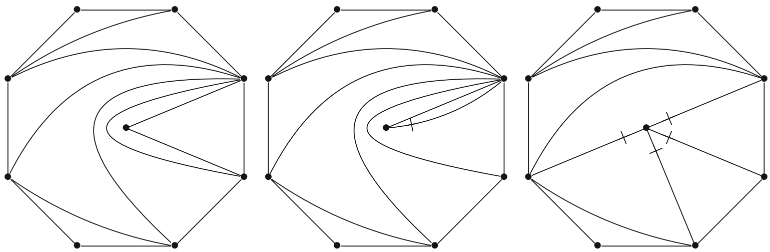


Fig. 3.11 Examples of triangulations

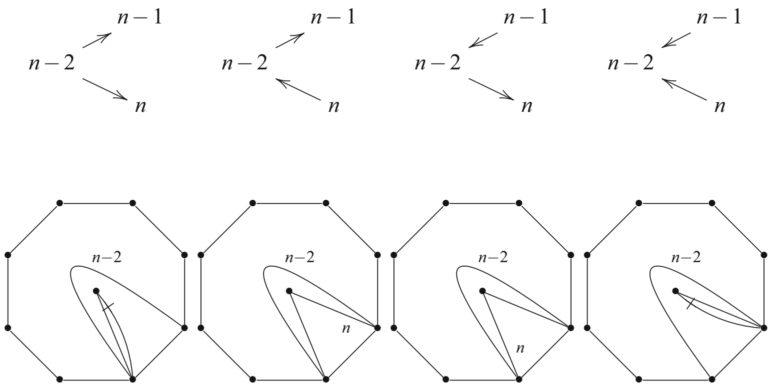
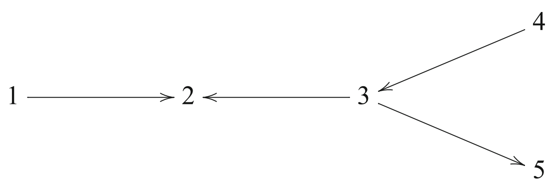


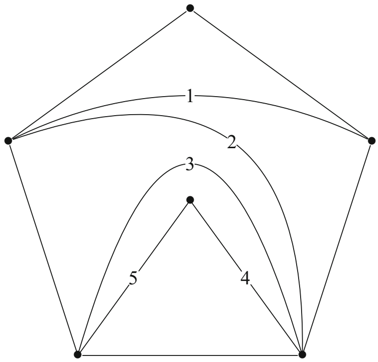
Fig. 3.12 Construction of the triangulation from the quiver

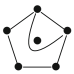
hand, $1 \to 2$ is in Q, then let γ_2 be the unique arc that forms a triangle Δ_1 together with γ_1 and a boundary segment in such a way that γ_1 is clockwise from γ_2 in Δ_1. Continue in this way until $n-2$ arcs are determined. For the arcs γ_{n-1} and γ_n which are corresponding to the vertices $n-1$ and n, respectively, there are four possibilities depending on the orientations of the arrows in the quiver; these four possibilities are displayed in Fig. 3.12.

In this way, the quiver



of Example 3.2 gives rise to the triangulation

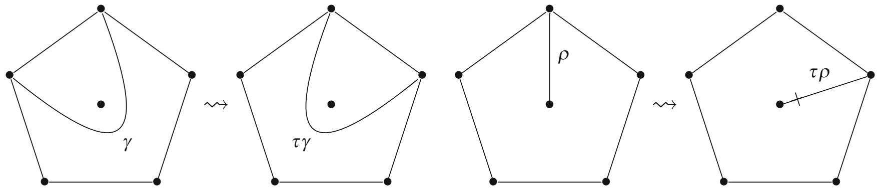


Since T_Q is a triangulation of the punctured polygon, any arc γ which is not already in T_Q will cut through a certain number of diagonals in T_Q; in fact, any such arc γ is uniquely determined by the set of diagonals in T_Q that γ crosses. To such a diagonal γ, we associate the indecomposable representation $M_\gamma = (M_i, \varphi_\alpha)$ of Q whose dimension at vertex i is given by the number of crossings $e(\gamma, \gamma_i)$ between the arc γ and the arc γ_i of the triangulation that corresponds to the vertex i of the quiver. In the example, the arc  crosses the arcs $1, 4, 5$ once and $2, 3$ twice, and the corresponding representation is isomorphic to

$$k \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix}1&0\\0&1\end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} k\,, \qquad k^2 \xrightarrow{\begin{bmatrix}0&1\end{bmatrix}} k$$

The map $\gamma \mapsto M_\gamma$ is a bijection from the set of arcs that are not in T_Q and the set of isoclasses of indecomposable representations of Q.

The Auslander–Reiten translation τ is given by an elementary clockwise rotation of the punctured polygon with simultaneous change of the tags at the puncture. So in our example



The projective representation $P(i)$ is given by τ^{-1} of the arc γ_i, and the injective representation $I(i)$ is given by τ of the arc γ_i. The complete Auslander–Reiten quiver can be easily constructed now, starting with the projectives and applying the elementary rotation to compute the τ-orbits until we reach the injective in each τ-orbit. The Auslander–Reiten quiver of Example 3.2 is shown in Fig. 3.13.

3.3.4 *Computing Hom Dimensions, Ext Dimensions, and Short Exact Sequences*

As in type $\mathbb{A}$, we can compute the dimensions of the Hom and Ext spaces easily from the Auslander–Reiten quiver in type $\mathbb{D}$.

3.3.4.1 Dimension of Hom(M, N)

Let Q be a type $\mathbb{D}$ quiver and let M, N be two indecomposable representations of Q. We can compute the dimension of the vector space $\mathrm{Hom}(M, N)$ using the relative position of M and N in the Auslander–Reiten quiver. The maximal slanted rectangles of type $\mathscr{A}$ have to be replaced by maximal *hammocks*. It is a little harder to describe these hammocks than the rectangles. Several examples are illustrated in Fig. 3.14.

Recall that a path $M_0 \to M_1 \to \cdots \to M_s$ in the Auslander–Reiten quiver is called a *sectional* path if $\tau M_{i+1} \neq M_{i-1}$ for all $i = 1, \dots, s-1$. As in type $\mathbb{A}$, we define $\Sigma_{\to}(M)$ to be the set of all indecomposable representations that can be reached from M by a sectional path and $\Sigma_{\leftarrow}(M)$ to be the set of all indecomposable representations from which one can reach M by a sectional path.

We can now construct the hammock by the following algorithm, refer to Fig. 3.14. Start by labeling each vertex in $\Sigma_{\to}(M)$ with the number 1. Then consider the almost split sequence $0 \to M \to E \to \tau^{-1}M \to 0$. Note that each summand of E lies in $\Sigma_{\to}(M)$ and that $\tau^{-1}M$ does not. Label the vertex $\tau^{-1}M$ by the number of indecomposable summands of E minus the label of M. Thus the label at $\tau^{-1}M$ is either 0, 1 or 2 depending on whether the mesh in the Auslander–Reiten quiver between M and $\tau^{-1}M$ has 1, 2 or 3 middle vertices, respectively.

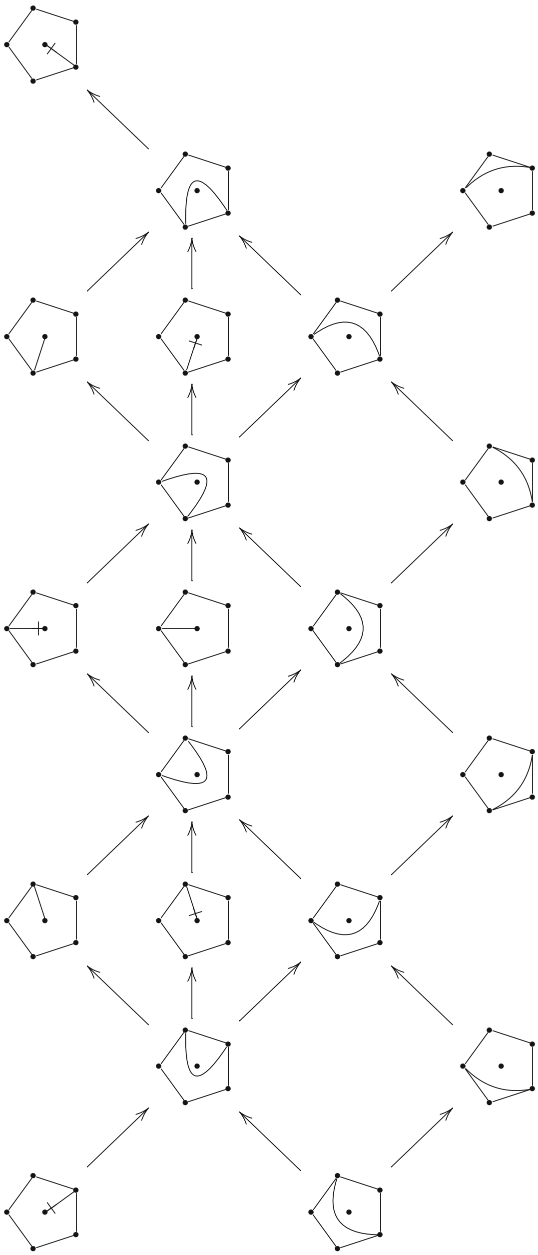


Fig. 3.13 Auslander–Reiten quiver of type $\mathbb{D}_5$ in terms of arcs in a punctured polygon

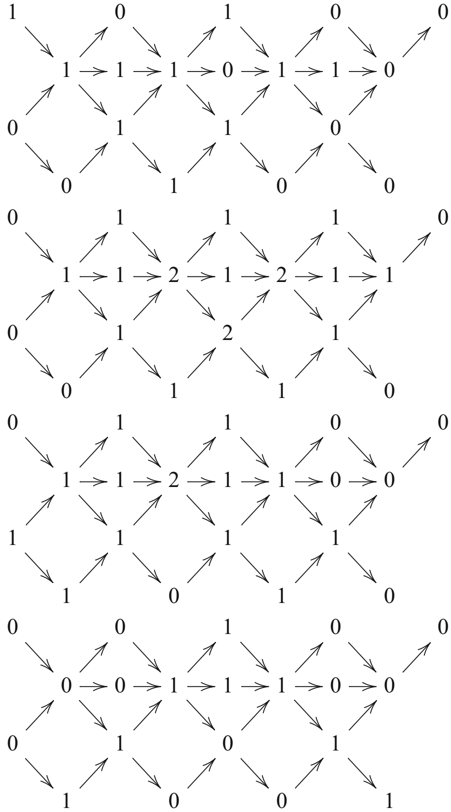


Fig. 3.14 Dimension of $\text{Hom}(M, -)$ for $M = P(5)$ on the *top left*, $M = P(3)$ on the *top right*, $M = P(2)$ on the *bottom left*, and $M = P(1)$ on the *bottom right*. The position of the representation M is at the leftmost 1 in each case; the numbers $0, 1, 2$ indicate the dimension of $\text{Hom}(M, N)$ for each indecomposable representation N

Recursively, for every almost split sequence $0 \to M' \to E' \to \tau^{-1}N' \to 0$ such that the vertices corresponding to M' and to each summand of E' are already labeled, define the label of the vertex corresponding to $\tau^{-1}M'$ to be the sum of the labels of the indecomposable summands of E' minus the label of M'. If this number is negative, then use the label 0 instead.

This labeling is called the hammock starting at M. If N is any indecomposable representation, then the label at the vertex corresponding to N is the dimension of $\text{Hom}(M, N)$. Thus these dimensions can be 0, 1 or 2.

Note that the same algorithm applied to an Auslander–Reiten quiver of type $\mathbb{A}$ will produce the maximal slanted rectangle $\mathscr{R}_{\to}(M)$. Note also that, as in type $\mathbb{A}$, the left boundary of the area with nonzero labels is $\Sigma_{\to}(M)$, and if $M = P(i)$ is an indecomposable projective, then the right boundary of the area with nonzero labels is $\Sigma_{\leftarrow}(I(i))$, and thus the hammock consists of all modules that are nonzero at the vertex i.

3.3.4.2 Ext^1 and Short Exact Sequences

We can compute Ext^1 as in type $\mathbb{A}$ thanks to the formula

$$\dim \operatorname{Ext}^1(M, N) = \dim \operatorname{Hom}(N, \tau M).$$

Thus the dimension of $\operatorname{Ext}^1(M, -)$ is determined by the maximal hammock ending at τM.

Since the dimension of $\operatorname{Ext}^1(M, N)$ can be as large as 2, it is not so easy to find the short exact sequences that represent the elements of $\operatorname{Ext}^1(M, N)$. We know that each element can be represented by short exact sequences of the form $0 \to N \to E \to M \to 0$, where E is some representation of Q, but there might be several choices for E. In the example in Fig. 3.15, there are four non-split short exact sequences starting at N and ending at M, namely

$$\begin{array}{l} 0 \to N \to E_1 \oplus E_2 \oplus H_2 \to M \to 0 \\ 0 \to N \to F_1 \oplus F_2 \oplus H_2 \to M \to 0 \\ 0 \to N \to \quad G_1 \oplus G_2 \quad \to M \to 0 \\ 0 \to N \to \quad H_1 \oplus H_2 \quad \to M \to 0. \end{array}$$

It is important to note that while there are four non-split short exact sequences, the dimension of $\operatorname{Ext}^1(M, N)$ is only two. Thus any two of the above sequences span the vector space $\operatorname{Ext}^1(M, N)$.

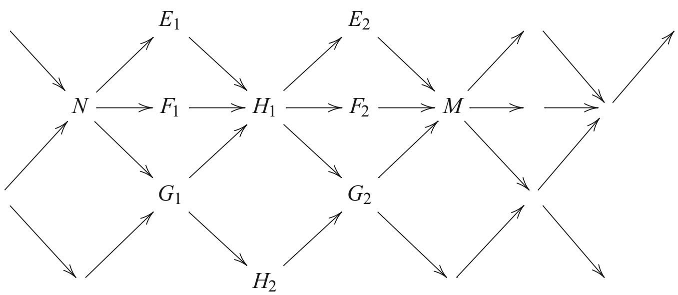
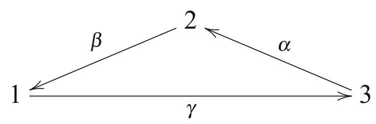


Fig. 3.15 Computing short exact sequences in type $\mathbb{D}$

3.4 Representations of Bound Quivers: Quivers with Relations

In this section, we want to study representations of quivers which, in contrast to earlier sections, are allowed to have oriented cycles or even loops. We had to exclude quivers with oriented cycles in Sects. 2.1–3.3 in order to be able to describe the indecomposable projective representation $P(i)$ at vertex i in terms of the paths that start at i. If the quiver has an oriented cycle that contains the vertex i, then there exist infinitely many paths that start at i, simply because we can run through the oriented cycle over and over again.

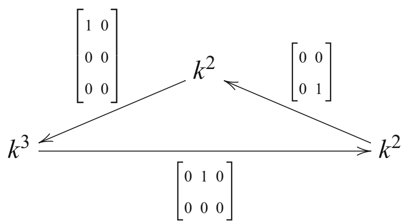
For this reason, we will only consider representations that satisfy certain relations given in terms of paths in the quiver. As an example, consider the quiver



There are infinitely many paths in Q, for example, those starting at vertex 3 include $e_3, \alpha, \alpha\beta, \alpha\beta\gamma, \alpha\beta\gamma\alpha, \alpha\beta\gamma\alpha\beta, \ldots$. We will allow only finitely many paths, by imposing certain *relations*, for example,

$$\alpha\beta = 0, \beta\gamma = 0 \text{ and } \gamma\alpha = 0.$$

Then there are only six nonzero paths, namely $e_1, e_2, e_3, \alpha, \beta$ and γ. Among the representations $M = (M_i, \varphi_\alpha)$ of Q we will then consider only those that satisfy the relations imposed on the quiver, which, in our example, means that $\varphi_\alpha \circ \varphi_\gamma = 0, \varphi_\beta \circ \varphi_\alpha = 0$, and $\varphi_\gamma \circ \varphi_\beta = 0$. For instance, the representation



satisfies these relations.

We will now formalize these ideas.

Definition 3.1. Let Q be a quiver.

1. Two paths c, c' in Q are called **parallel** if $s(c) = s(c')$ and $t(c) = t(c')$.
2. A **relation** ρ is a linear combination $\rho = \sum_c \lambda_c c$ of parallel paths each of which has length at least two.

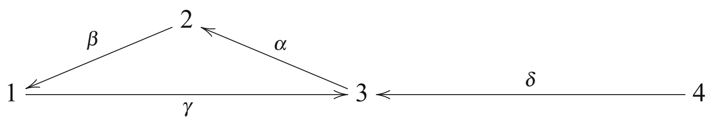
3. A **bound quiver** (Q, R) is a quiver Q together with a set of relations R.

Definition 3.2. Let (Q, R) be a bound quiver. A **representation of** (Q, R) is a representation $M = (M_i, \varphi_\alpha)$ of Q such that $\varphi_\rho = 0$, for each relation $\rho \in R$, where $\varphi_\rho = \sum_c \lambda_c \varphi_c$ if $\rho = \sum_c \lambda_c c$.

Define rep (Q, R) to be the category of representations of (Q, R). We can define morphisms, direct sums, kernels, and cokernels in the same way as in rep Q. The simple representations $S(i)$ are defined in the same way as in rep Q.

To define the indecomposable projective and the indecomposable injective representations, we need the notion of path algebra which we will define in Chap. 4. For now, let us content ourselves with some examples.

Let Q be the quiver



and let $R = \{\alpha\beta, \beta\gamma, \gamma\alpha\}$. Then the paths in the bound quiver (Q, R) are $e_1, \gamma, e_2, \beta, e_3, \alpha, e_4, \delta, \delta\alpha$, and the indecomposable projective representations are

$$P(1) = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \quad P(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \quad P(3) = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \quad P(4) = \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix}.$$

Note that the category rep (Q, R) is not hereditary. Indeed, the simple representation $S(3) = 3$ has the following minimal projective resolution

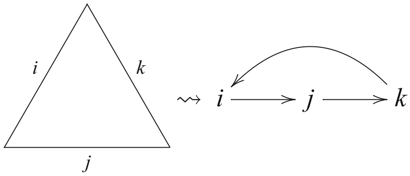
$$\cdots \longrightarrow \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \longrightarrow 3 \longrightarrow 0$$

which does not stop after two steps.

3.4.1 Cluster-Tilted Bound Quivers of Type $\mathbb{A}_n$

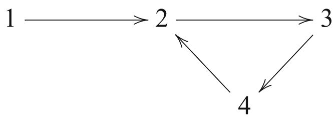
In Sect. 3.1.3, we have used triangulations of a polygon with $n + 3$ vertices to construct the Auslander–Reiten quiver of the type A quivers. Note however that the triangulations we used then had the property that each triangle has at least one side on the boundary of the polygon. The cluster-tilted quivers of type $\mathbb{A}_n$ are precisely those that are associated to an arbitrary triangulation of the $(n + 3)$-gon.

Let $T = \{1, 2, \ldots, n\}$ be a triangulation of a polygon with $n+3$ vertices. Define a quiver $Q = (Q_0, Q_1)$ by $Q_0 = T$, and there is an arrow $i \to j$ in Q_1 precisely if the diagonals i and j bound a triangle in which j lies counterclockwise of i:

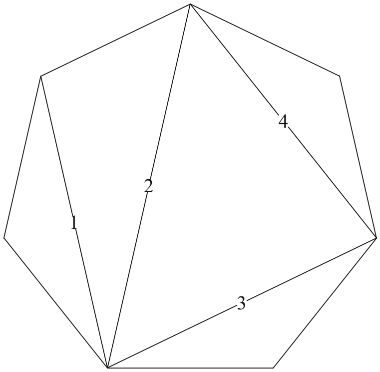


Define the set of relations R to be the set of all paths $i \to j \to k$ such that there exists an arrow $k \to i$. The Auslander–Reiten quiver of (Q, I) can be constructed using diagonals in a polygon with $(n+3)$ vertices in exactly the same way as for the path algebras of type $\mathbb{A}_n$.

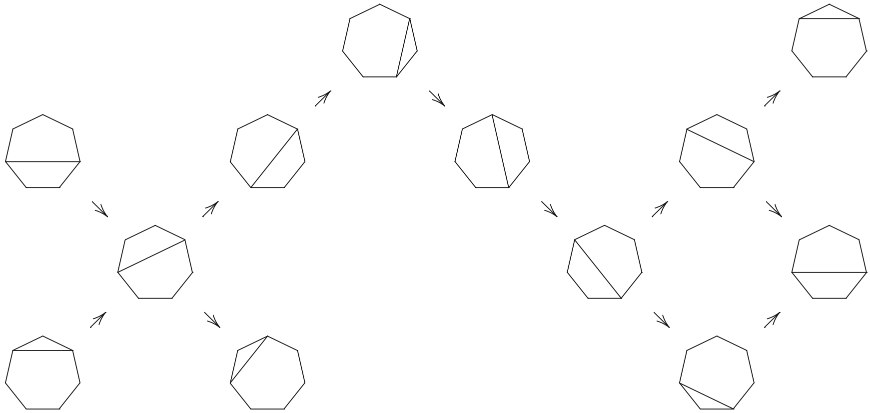
We illustrate this method in an example. Let Q be the quiver



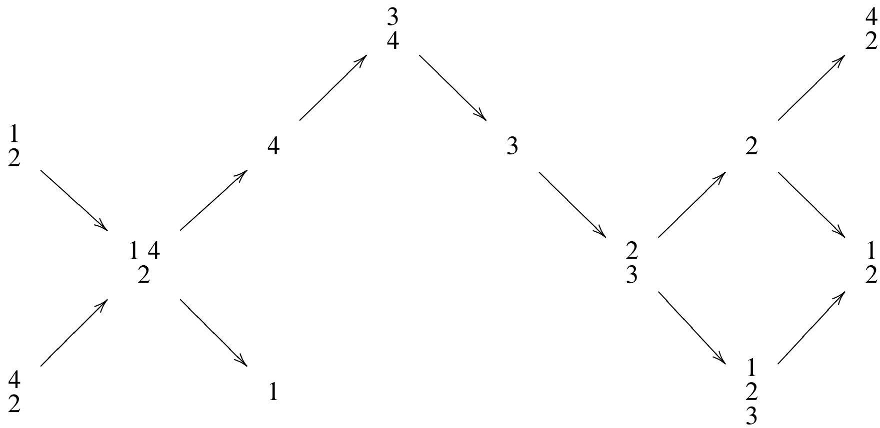
associated to the triangulation



Then the Auslander–Reiten quiver is



which translates into



where one has to identify the two representations labeled $\begin{smallmatrix}1\\2\end{smallmatrix}$ and the two representations labeled $\begin{smallmatrix}4\\2\end{smallmatrix}$, so that the Auslander–Reiten quiver has the shape of a Moebius strip.

Note that the number of indecomposable representations of Q is equal to the number of all diagonals in an $(n + 3)$-gon minus the n diagonals in the given triangulation.

Let us compute the number of diagonals. For every vertex a of the polygon, the diagonals starting at a may end at any vertex of the polygon except at a and at its two neighbors. So there are n diagonals starting at each vertex a. There are $n + 3$ possibilities for the vertex a, but when we consider them all, we count each diagonal exactly twice. Therefore the number of diagonals is $n(n + 3)/2$.

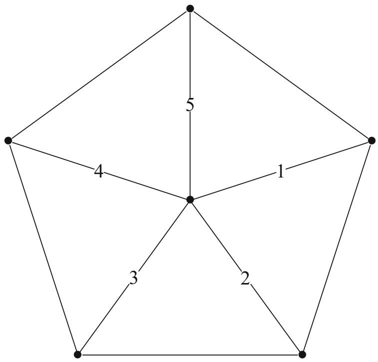
Now the number of indecomposable representations of Q is equal to $n(n+3)/2-n$ which is equal to $n(n + 1)/2$.

In particular the cluster-tilted quivers of type $\mathbb{A}_n$ and the quivers of type $\mathbb{A}_n$ have the same number of indecomposable representations.

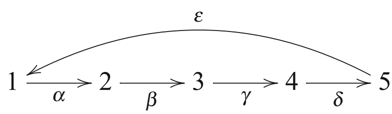
3.4.2 Cluster-Tilted Bound Quivers of Type $\mathbb{D}_n$

In Sect. 3.3.3, we have used triangulations of a punctured polygon to compute the Auslander–Reiten quiver of type $\mathbb{D}_n$ quivers. The triangulations we considered then all had the property that there were always exactly two arcs incident to the puncture and that every triangle in the triangulation had at least one edge on the boundary. The **cluster-tilted** quivers of type $\mathbb{D}_n$ are precisely those that are associated to an arbitrary triangulation of the punctured n-gon. The quiver is determined from the triangulation just as in Sect. 3.3.3.

For example, the triangulation



gives rise to the quiver



bound by the relations $\alpha\beta\gamma\delta = \beta\gamma\delta\epsilon = \gamma\delta\epsilon\alpha = \delta\epsilon\alpha\beta = \epsilon\alpha\beta\gamma = 0$; and its Auslander–Reiten quiver is given in terms of arcs in Fig. 3.16 and in terms of representations in Fig. 3.17.

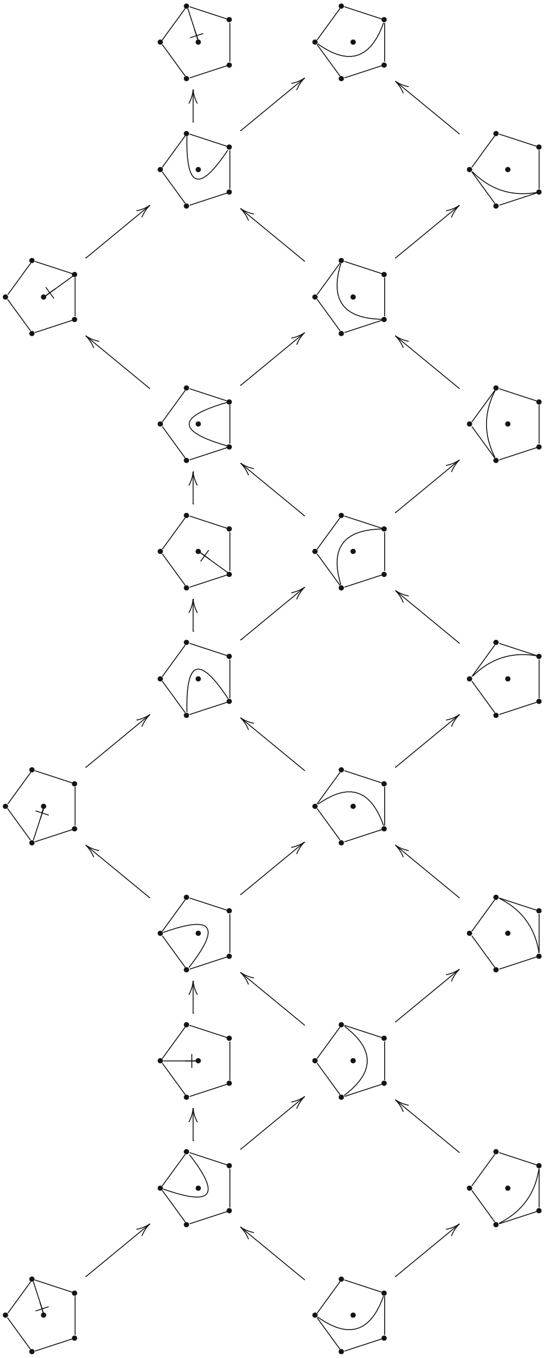


Fig. 3.16 Auslander–Reiten quiver of cluster-tilted type $\mathbb{D}_5$ in terms of arcs. The two vertices on the *far left* are to be identified with the two vertices on the *far right*

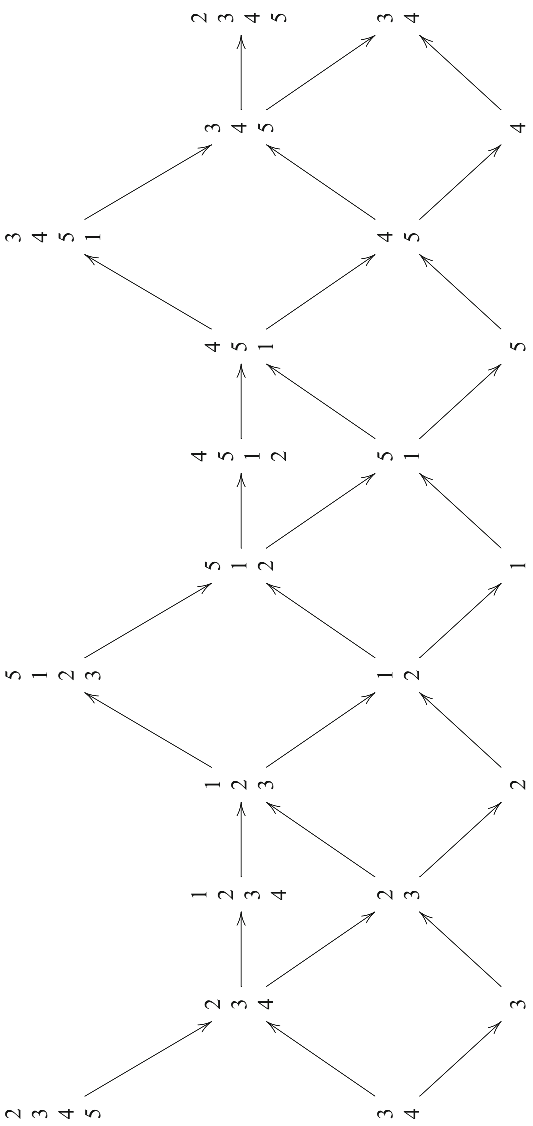


Fig. 3.17 Auslander–Reiten quiver of cluster-tilted type $\mathbb{D}_5$ in terms of representations. The two vertices on the *far left* are to be identified with the two vertices on the *far right*

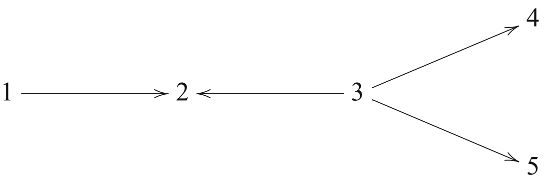
3.5 Notes

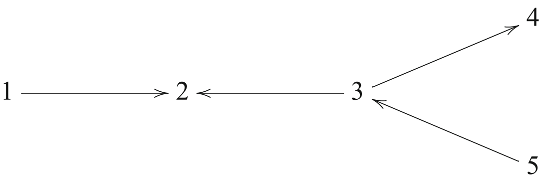
Further information on the construction of Auslander–Reiten quivers can be found in [8, 35]; more on representation type and Gabriel's Theorem in [18, 30, 33]. The construction of Auslander–Reiten quivers from triangulations was introduced in [28, 54].

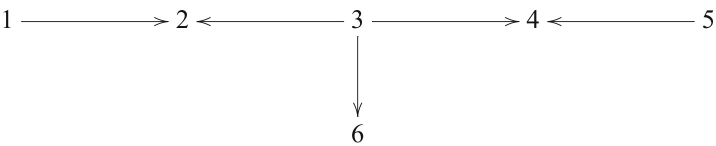
Problems

Exercises for Chap. 3

3.1. Compute the Auslander–Reiten quivers of the following quivers:

1. $1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \longleftarrow 6$
2. $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longleftarrow 6$
3. 

4. 

5. 

3.2. Let Q be the quiver

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \longleftarrow 6$$

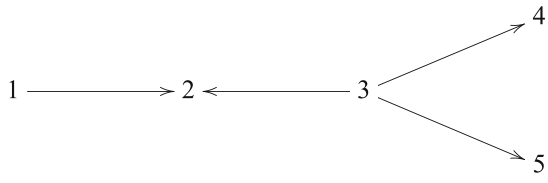
and consider the indecomposable representations L and N given by the dimension vectors

$$\underline{\dim}\, L = (0,1,1,1,0,0) \text{ and } \underline{\dim}\, N = (0,0,1,1,1,1).$$

Prove that $\dim \operatorname{Ext}^1(N, L) = 1$ and find the middle term of a non-split short exact sequence of the form

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

3.3. Let Q be the quiver



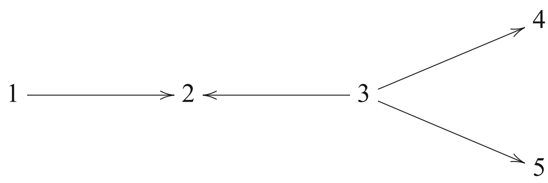
and consider the indecomposable representations L and N given by the dimension vectors $\underline{\dim}\, L = (0,1,1,1,1)$ and $\underline{\dim}\, N = (1,1,1,0,0)$.

1. Prove that $\dim \operatorname{Ext}^1(N, L) = 2$ and find 4 non-equivalent non-split short exact sequences of the form

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

2. Show that L is projective and that N is injective.
3. Show that $\tau^3 N$ is a summand of the radical of L.

3.4. Let Q be the quiver



and consider the indecomposable representations $L = \begin{smallmatrix} & 3 & \\ 2 & & 4 \end{smallmatrix}$ and $N = \begin{smallmatrix} 1 & & 3\,3 & \\ & 2 & 4 & 5 \end{smallmatrix}$.

1. Prove that there is a unique representation M for which there exists a non-split short exact sequence

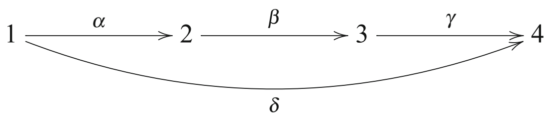
$$0 \longrightarrow L \longrightarrow M \xrightarrow{\;f\;} N \longrightarrow 0\ .$$

2. Let $M' = \begin{smallmatrix} 1\,3 \\ 2\,5 \end{smallmatrix}$ and $g : M' \to N$ be the inclusion morphism. Then the fiber product X of f and g, defined in Exercise 1.8 of Chap. 1, gives a short exact sequence

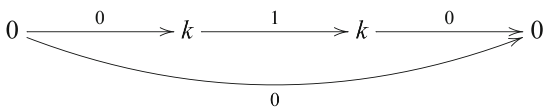
$$0 \longrightarrow L \longrightarrow X \longrightarrow M' \longrightarrow 0\ .$$

Prove that $L = \tau M'$.

3.5. Let Q be the quiver

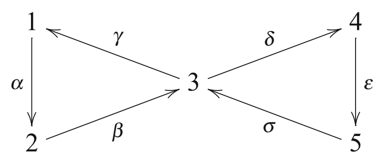


and let M be the indecomposable representation



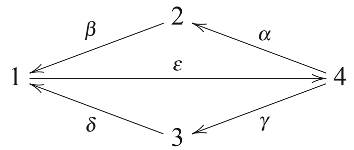
Compute $L_1 = \tau M$, $L_2 = \tau^2 M$ and $L_3 = \tau^3 M$ using the Nakayama functor. Find three representations N_1, N_2 and N_3, by explicitly writing out the matrices, such that $\underline{\dim} N_i = (1, 1, 1, 1)$ and L_i is a subrepresentation of N_i, for $i = 1, 2, 3$.

3.6. Compute the Auslander–Reiten quiver of (Q, I), where Q is the quiver



and $I = \{\alpha\beta, \beta\gamma, \gamma\alpha, \delta\epsilon, \epsilon\sigma, \sigma\delta\}$. [Hint: Use a triangulated polygon.]

3.7. Compute the Auslander–Reiten quiver of (Q, I), where Q is the quiver



and $I = \{\alpha\beta - \gamma\delta, \epsilon\alpha, \epsilon\gamma, \beta\epsilon, \delta\epsilon\}$. [Hint: Use a triangulated punctured polygon.]