Dihedral Sieving on Cluster Complexes

Zachary Stier, Julian Wellman, Zixuan Xu

Princeton University, Massachusetts Institute of Technology, Massachusetts Institute of Technology

Cyclic Sieving

Definition ([RSW04])

Suppose the cyclic group $C_n = \langle r \rangle$ acts on a finite set X, and $X(q)$ is a polynomial in q. The pair $(X \circlearrowleft C_n, X(q))$ has the *cyclic sieving phenomenon* (CSP) if, for all $\ell \in [n]$,

$$|\{x \in X : r^\ell x = x\}| = X(e^{2\ell\pi i/n}).$$

Equivalently, the number of points fixed by $r^\ell \in C_n$ can be computed by evaluating a certain polynomial at an appropriate nth root of unity. Think of this as the eigenvalue of r^ℓ in the representation of C_n which sends r to a $1/n$-th rotation on $\mathbb{C}$. This point of view allows us to generalize sieving to any group.

Sieving Phenomena for any Group

Definition

Suppose a group G acts on finite set X, and $X(q_1, \ldots, q_d)$ is a symmetric polynomial in d variables with ρ a d-dimensional representation of G. The pair $(X \circlearrowleft G, \rho, X(q_1, \ldots, q_d))$ exhibits *G-sieving* if, for all $g \in G$, if $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $\rho(g)$, then

$$|\{x \in X : gx = x\}| = X(\lambda_1, \ldots, \lambda_d).$$

In the case of the dihedral group $I_2(n) := \langle r, s | r^n = s^2 = 1,\ rs = sr^{-1} \rangle$, we use the representation ρ_{def} which sends r to a rotation by $2\pi/n$ radians and s to a reflection about the x-axis.

Odd Dihedral Sieving

Definition (cf. [RS17, Proposition 4.3])

Suppose the dihedral group $I_2(n)$ with n odd acts on a finite set X, and $X(q,t)$ is a symmetric polynomial in q and t. The pair $(X \circlearrowleft I_2(n), X(q,t))$ has the *dihedral sieving phenomenon* (DSP) if, for all $g \in I_2(n)$ with eigenvalues $\{\lambda_1, \lambda_2\}$ for $\rho_{\text{def}}(g)$,

$$|\{x \in X : gx = x\}| = X(\lambda_1, \lambda_2).$$

Rao and Suk [RS17] found many examples of odd dihedral sieving using a different, but equivalent definition.

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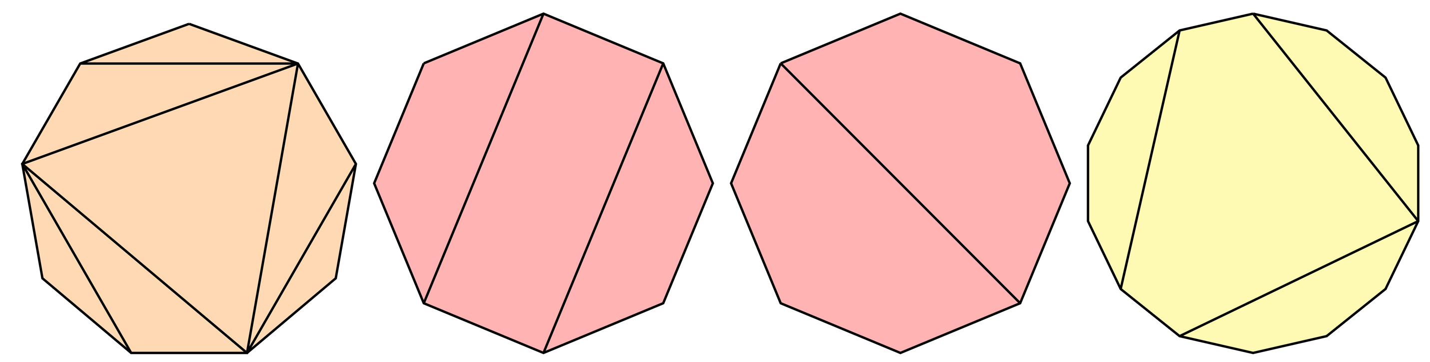
One set X for which Rao and Suk found nice dihedral sieving polynomials for is the set of triangulations of an n-gon under the natural dihedral action of rotation and reflection. These polynomials (denoted $\text{Cat}_n(q,t)$) are a (q,t)-analogue of the usual Catalan numbers $\text{Cat}_n = \frac{1}{n+1} \cdot \binom{2n}{n}$, which count (among many other things) the number of triangulations of an $(n+2)$-gon. We generalized this result in two different directions.

Results



Examples of k-angulations with rotational and/or reflective symmetry.

Theorem

Let X be the set of k-angulations of an n-gon, for odd $n \equiv 2 \pmod{k-2}$. Then the pair $(X \circlearrowleft I_2(n), \text{FussCat}_{k-2,\frac{n-2}{k-2}}(q,t))$ exhibits dihedral sieving.

Here, $\text{FussCat}_{s,m}(q,t) := \text{Cat}_{ms+1,m}(q,t)$ is a special case of the *rational* (q,t)-Catalan polynomials. When $t = 1$, this polynomial is the area-generating function for Dyck paths lying above the line connecting $(0,0)$ and (a,b).

Triangulations, pentagulations, Catalan numbers, and Fuss–Catalan numbers are all examples of *Type A* phenomena. Compatible clusters of almost-positive roots in Type A have a natural reflection action, which turns out to be isomorphic to the action of reflection on triangulations. A similar phenomenon occurs in root systems of other types [FR05]. In the following theorem, $\text{Cat}(\Phi, q, t)$ are the (q,t)-Catalan polynomials of type Φ, as defined by Stump [Stu10].

Theorem

Suppose $\Delta(\Phi)$ is the cluster complex for type Φ, with $I_2(n)$ action generated by certain involutions τ_+, τ_- on almost-positive roots. The pair $(\Delta(\Phi) \circlearrowleft I_2(n), \text{Cat}(\Phi, q, t))$ exhibits dihedral sieving for all odd n and Φ of type A, B/C, D, E, F, or I.

References

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Raney Numbers

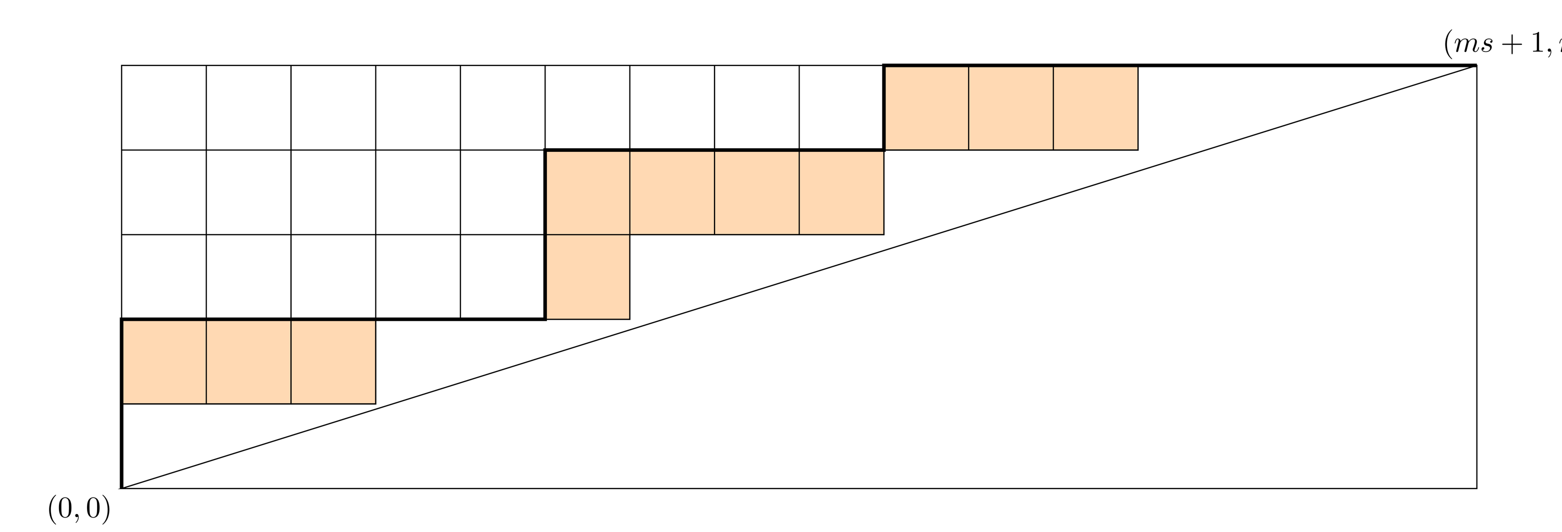
To prove an instance of dihedral sieving, we usually count the number of elements of the set X fixed by each dihedral group element combinatorially, and check that the result matches with the evaluation of the polynomial. In most cases, cyclic sieving results of Eu and Fu [EF08] allow us to focus on the group elements representing reflections. A nice bijection with *coral diagrams* allows us to show that the number of k-angulations which share a certain axis of symmetry is a certain *Raney number*. The Raney numbers are defined as

$$R_{p,r}(k) := \frac{r}{kp+r}\binom{kp+r}{k}.$$

Interestingly, these numbers specialize to the Fuss–Catalan numbers when $r = 1$, and the usual Catalan numbers when $(p,r) = (2,1)$. In this case, evaluating the polynomial turned out to be the trickier part. We needed to show:

$$\text{Cat}_{ms+1,m}(-1,1) := \sum_P (-1)^{\text{area}(P)} = R_{s+1,\frac{s+1}{2}}\binom{m-1}{2}.$$

Here the sum is over all Dyck paths P connecting $(0,0)$ to $(ms+1,m)$ while staying above the non-main diagonal, as pictured below.



Young diagram for $m = 5$ and $s = 3$, with an area cut out by a Dyck path.

The proof of this equality led us to discover an interesting determinant identity and rediscover some recursions on Raney numbers.

Future Work

1. We generalized dihedral sieving for triangulations in two directions; first by introducing the Fuss parameter, and second by considering other types. One could hope to prove a single unifying result by showing dihedral sieving for non-maximal clusters of arbitrary type. Unfortunately, the properties of the (q,t)-Fuss–Catalan polynomials are still quite conjectural in types other than A, and the objects in question only get harder to work with. Other potential sets to look at for dihedral sieving include k-divisible dissections and the rational associahedron.

2. Our results apply only to the case of odd n, because the representation theory of $I_2(n)$ is more complicated when n is even. In particular, there is an additional conjugacy class, which gives rise to additional irreducible representations. We have some ideas for how to deal with these challenges, but more work is needed to determine whether a natural notion of dihedral sieving even exists for n even.

3. Though it is not discussed in this poster, we have also introduced the sieving phenomenon for the symmetric group, and shown an instance of symmetric sieving for multisets with complete homogeneous polynomials. We also gave a conjecture about symmetric sieving for subsets, using Schur polynomials, which was subsequently proven by Christopher Ryba. It would be very interesting to see more examples of symmetric sieving.