FUNCTORIALITY OF CRITICAL GROUPS

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Abstract. We offer a categorical presentation of the critical group of a graph. We take the lattice-theoretic approach to critical groups developed in [1], and consider the extent to which this construction is functorial. We build a category $\mathcal{R}OD$, and construct a functor Crit from $\mathcal{R}OD$ that takes values in abelian groups with $\mathbb{Q}/\mathbb{Z}$-valued pairings. For an appropriate category of graphs, we show that the construction of the critical group is a functor that factors through $\mathcal{R}OD$. In particular we show that certain kinds of simplicial maps between graphs, for example the projection map of a covering space, induce homomorphisms between the critical groups. Along the way we phrase certain old results about critical groups in these terms.

1. Introduction

Let Γ be a finite graph, and for a ring R let RC_i be the group of i-chains with coefficients in R. Taking R to be $\mathbb{R}$, these chain groups are real vector spaces, and they have a fairly natural Euclidean inner product by taking the i-cells to be an orthonormal basis. Taking R to be $\mathbb{Z}$, $\mathbb{Z}C_i \subset \mathbb{R}C_i$ has the structure of an *integral lattice*: the inner product takes values in $\mathbb{Z}$ when restricted to integral chains.

There are two subspaces of $\mathbb{R}C_i$: the subspace of 1-cycles (sometimes called flows), and the subspace of 1-coboundaries (sometimes called cuts). With respect to the inner product just mentioned, these subspaces are an orthogonal decomposition of the space of chains. Then we have two lattices within these subspaces, of integral cycles (what we call $.Z$) and of integral coboundaries ($\dot{B}$). It was pointed out in [1] that these lattices carry interesting information about the graphs. In particular, the determinant group of either one of these lattices is isomorphic to the *critical group* [4, 14.13] of the graph.

In this paper we use the theory developed in [1] to give the Critical group construction the structure of a functor. We give the following applications:

There is a notion of "divisibility" for graphs. A graph Γ divides a graph Γ' if there is a map $p : \Gamma' \to \Gamma$ satisfying certain properties; they are satisfied for example if p is the projection of a covering space. We call the data $p : \Gamma' \to \Gamma$ a "Berman bundle." All this was defined implicitly in [2], where it was shown that if Γ divides Γ' then the tree number of Γ divides the tree number of Γ'. We strengthen this result to a statement about the structure of the Critical groups of Γ and Γ'.

We may divide each edge of a graph Γ into k edges, giving us a graph Γ_k. For example if C is the n-gon or n-cycle, C_k is the $n \times k$-gon. It was proven in [5] that the Critical groups of Γ and Γ_k are related by a short exact sequence:

$$0 \to (\mathbb{Z}/k\mathbb{Z})^m \to \mathrm{Crit}(\Gamma_k) \to \mathrm{Crit}(\Gamma) \to 0$$

We interpret this result in terms of the Crit functor.

2. The Functor Crit

We will be considering Euclidean vector spaces and lattices inside of them, typically denoted by $\mathbb{Z}X \subset \mathbb{R}X$. If there is no danger of confusion (or if we are just being coy) we will write simply X for either $\mathbb{Z}X$ or $\mathbb{R}X$. All latices have full rank.

We will be considering functors of either variance. If X is a functor or aids in constructing a functor or in any way can be thought of as having a variance, then we write $.X$ if it is to be thought of covariantly and ${}^{\cdot}X$ if it is to be thought of contravariantly. This is inspired by a similar practice in homological algebra where things are written as $X.$ or $X^{\cdot}$, but here this implies that X has a grading, which is not true of $.X$. This may be of no mathematical significance but I find it psychologically useful.

Definition 1. A *rational orthogonal decomposition*, or a ROD, consists of the following data:

- A Euclidean vector space $\mathbb{R}C$,
- An orthonormal lattice $\mathbb{Z}C \subset \mathbb{R}C$
- An orthogonal decomposition $\mathbb{R}C = \mathbb{R}.Z \oplus \mathbb{R}^{\cdot}B$

We require that $\mathbb{R}.Z$ and $\mathbb{R}^{\cdot}B$ be rational with respect to the lattice $\mathbb{Z}C$. We write $\mathbb{Z}.Z$ for $\mathbb{Z}C \cap \mathbb{R}.Z$ and similarly for $\mathbb{Z}^{\cdot}B$. We usally just write C for a ROD.

Note that $.Z = \mathbb{Z}.Z$ and ${}^{\cdot}B$ are integral lattices. (A lattice Λ is *integral* if $\langle \Lambda, \Lambda \rangle \subset \mathbb{Z}$.) Elements of the lattice are called "integral". Elements of $.Z$ are "cycles" and elements of ${}^{\cdot}B$ are "coboundaries". Typically we will define

$$n = \dim(C) = \dim(\mathbb{R}C) = \dim(\mathbb{Z}C)$$

$$m = \dim(.Z) \qquad \mu = \dim({}^{\cdot}B)$$

Thus $m + \mu = n$.

Example 2. Let C, C' be Euclidean vector spaces each containing an orthonormal lattice. Let $\partial : C \to C'$ and $\partial^* : C' \to C$ be adjoint linear maps, defined over the rationals with respect to the lattice. Then $\mathbb{R}C = \ker(\partial) \oplus \mathrm{im}(\partial^*)$. We define $.Z = \ker(\partial)$ and ${}^{\cdot}B = \mathrm{im}(\partial^*)$. Then C is a ROD. Most examples of RODs will be associated to an adjoint pair of linear map in this way.

Example 3. Let $\mathbb{R}C_i$, $i = 0$ or 1 be the space of real i-chains and $\mathbb{Z}C_i \subset \mathbb{R}C_i$ be the integral chains of a graph Γ. Then C_i has a natural inner product which is the Kronecker delta on i-cells (vertices or edges). We have the boundary operator $\partial : C_1 \to C_0$ and the coboundary operator $\partial^* : C_0 \to C_1$, which are adjoint linear maps. Then by the previous example C_1 is a ROD.

Definition 4. We define a category pre-$\mathcal{ROD}$, whose objects are RODs and whose morphisms are linear maps $f : \mathbb{R}C \to \mathbb{R}C'$ that satisfy $f(\mathbb{Z}C) \subset \mathbb{Z}C'$ and $f^*(\mathbb{Z}C') \subset \mathbb{Z}C$. We don't require that f preserve $.Z$ or ${}^{\cdot}B$. There is a contravariant involution on pre-$\mathcal{ROD}$ given by taking f to its adjoint f^*.

Not all maps $\mathbb{Z}C \to \mathbb{Z}C'$ have an adjoint, but every map induces a unique map $\mathbb{R}C \to \mathbb{R}C'$. We could thus define a map $C \to C'$ to be a pair of adjoint linear maps (f, f^*) between $\mathbb{Z}C$ and $\mathbb{Z}C'$.

Proposition 5. Let $f : C \to C'$ be a map in pre-$\mathcal{ROD}$

(i) $f(.Z) \subset .Z' \iff f^*({}^{\cdot}B') \subset {}^{\cdot}B$
(ii) $f^*(.Z') \subset \cdot Z \iff f({}^{\cdot}B) \subset {}^{\cdot}B'$

Proof. The proofs of each of the four implications are virtually identical. We only prove one direction of part (i). Suppose

$$f(.Z) \subset .Z'$$

Let $x \in {}^{\cdot}B'$ and $y \in .Z$. Then $\langle f^*(x), y\rangle = \langle x, f(y)\rangle$ and since $f(y) \in .Z'$ and $\langle .Z', {}^{\cdot}B'\rangle = 0$ we have $\langle f^*(x), y\rangle = 0$ for every $y \in .Z$. So $f^*(x)$ is in the orthogonal complement of $.Z$, *i.e.* it is in ${}^{\cdot}B$. □

A map that satisfies (i) (resp. (ii)) is called type (i) (resp. type (ii)). Note the following facts: The composition of two type (i) (resp. type (ii)) maps is type (i) (resp. type (ii)). The adjoint of a type (i) (resp. type (ii)) map is type (ii) (resp. type (i)).

Definition 6. We define $\mathcal{R}OD$ to be the subcategory of pre-$\mathcal{R}OD$ whose objects are the same and whose morphisms are maps that are both type (i) and type (ii). The involution on pre-$\mathcal{R}OD$ restricts to one on $\mathcal{R}OD$, by the preceding remarks.

Let V be a Euclidean vector space and Λ an integral lattice inside of it. The *dual lattice* $\Lambda^\#$ is defined by $\{x \in V : \langle x, \Lambda\rangle \subset \mathbb{Z}\}$. Since Λ is integral, $\Lambda \subset \Lambda^\#$, and we write $\det(\Lambda)$ for $\Lambda^\#/\Lambda$. Of course $\langle \Lambda^\#, \Lambda^\#\rangle \subset \mathbb{Q}$, so we get an induced inner product on $\det(\Lambda)$;

$$\langle , \rangle_{\mathbb{Q}/\mathbb{Z}} : \det(\Lambda) \otimes \det(\Lambda) \to \mathbb{Q}/\mathbb{Z}$$

The abelian group $\det(\Lambda)$ is called the "determinant group" of Λ. It's order is the volume squared of a fundamental region. $\langle , \rangle_{\mathbb{Q}/\mathbb{Z}}$ is nonsingular.

If one doesn't care about the inner product of $\Lambda^\#$, then we may identify it with $\mathrm{Hom}(\Lambda, \mathbb{Z})$, and we have the exact sequence

$$0 \to \Lambda \to \mathrm{Hom}(\Lambda, \mathbb{Z}) = \Lambda^\# \to \det(\Lambda) \to 0$$

.

Definition 7. $\mathcal{A}$ is the category of finite abelian groups with nonsingular inner products. $\mathcal{A}$ has a contravariant involution, the "Pontrjagin dual" given by

$$a \mapsto \mathrm{Hom}(a, \mathbb{Q}/\mathbb{Z}).$$

See [7, Chapter 5].

Now, suppose we have lattices $\Lambda \subset V$ and $M \subset W$. We want to know which linear maps $f : V \to W$ induce a map $f : \det(\Lambda) \to \det(M)$. Clearly we need $f(\Lambda) \subset M$ and $f(\Lambda^\#) \subset M^\#$.

Proposition 8. Let $f : C \to C'$ be a map in pre-$\mathcal{R}OD$. Then we have

(i) $f(\mathbb{Z}.Z) \subset \mathbb{Z}.Z' \iff f^*(\mathbb{Z}^{\cdot}B') \subset \mathbb{Z}^{\cdot}B \iff f$ is type (i).

(ii) $f(\mathbb{Z}.Z)^\# \subset (\mathbb{Z}.Z')^\# \iff f^*(\mathbb{Z}^{\cdot}B')^\# \subset (\mathbb{Z}^{\cdot}B)^\# \iff f$ is type (ii).

Proof. (i) Obvious in view of 5.

(ii) Suppose f is type (ii). Let $x \in (\mathbb{Z}.Z)^\#$ and let $y \in \mathbb{Z}.Z'$. Then $\langle f(x), y\rangle = \langle x, f^*(y)\rangle \in \mathbb{Z}$. So $f(x) \in (\mathbb{Z}.Z')^\#$.

□

Thus, if f is type (i) and type (ii), there is an induced map $f : \det(.Z) \to \det(.Z')$ and a map $f : \det({}^{\cdot}B') \to \det({}^{\cdot}B)$. So we have two functors from $\mathcal{R}OD$ to $\mathcal{A}$, one covariant and one contravariant. Explicitly, we have $x + .Z \mapsto f(x) + .Z'$ and

$x + {}^{\cdot}B' \mapsto f^*(x) + {}^{\cdot}B$. Note that if we take the induced map of f^* on $\det({}^{\cdot}B)$, we have $x + {}^{\cdot}B \mapsto f(x) + {}^{\cdot}B'$.

Proposition 9. The following diagrams commute up to natural isomorphism:

$$\begin{array}{ccc} \mathcal{ROD} & \xrightarrow{\det(.Z)} & \mathcal{A} \\ {\scriptstyle involv.}\downarrow & & \| \\ \mathcal{ROD}^o & \xrightarrow[\det({}^{\cdot}B)]{} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} \mathcal{ROD} & \xrightarrow{\det(.Z)} & \mathcal{A} \\ \| & & \downarrow{\scriptstyle involv.} \\ \mathcal{ROD} & \xrightarrow[\det(.Z)]{} & \mathcal{A}^o \end{array}$$

The vertical arrows are the contravariant involutions described in the definitions.

Proof. This proof is essentially [1, Lemma 1], although it is not stated in these terms. Consider the orthogonal projection maps $p_Z : C \to .Z$ and $p_B : C \to {}^{\cdot}B$. It is easily checked that $p_Z(\mathbb{Z}C) = (\mathbb{Z}.Z)^{\#}$ and that $p_B(\mathbb{Z}(C)) = (\mathbb{Z}{}^{\cdot}B)^{\#}$. We can compose the maps p with the projections $\pi : \Lambda^{\#} \to \det(\Lambda)$. That is, we have:

$$\det(.Z) \xleftarrow{\pi \circ p_Z} \mathbb{Z}C \xrightarrow{\pi \circ p_B} \det({}^{\cdot}B)$$

The kernel of both maps is $.Z \oplus {}^{\cdot}B$. Thus we have isomorphisms:

$$\det(.Z) \xleftarrow{\sim} \mathbb{Z}C/(.Z \oplus {}^{\cdot}B) \xrightarrow{\sim} \det({}^{\cdot}B)$$

If we take the inverse of the right map, we have:

$$\det(.Z) \xrightarrow[\sim]{} \det({}^{\cdot}B)$$

We claim that this is the natural isomorphism that makes both of the diagrams commute.

Let $f : C \to C'$. Note that f commutes with p_Z and with p_B. We need commutativity of

$$\begin{array}{ccc} \det(.Z) & \xrightarrow{f} & \det(.Z') \\ \sim\downarrow & & \downarrow\sim \\ \det({}^{\cdot}B) & \xrightarrow[f^*]{} & \det({}^{\cdot}B') \end{array}$$

where the lower horizontal map is the induced map from the involution of f (= the involution applied to the induced map from f. $x + {}^{\cdot}B \mapsto f(x) + {}^{\cdot}B'$)

Let $x \in (.Z)^{\#}$, $y \in C$ and $y' \in C'$ have $p_Z(y) = x$ and $p_Z(y') = f(x)$. We can take $y' = f(y)$.

Then we get

$$x + .Z \mapsto f(x) + .Z' \mapsto y' + (.Z' \oplus {}^{\cdot}B') \mapsto p_B(y') + {}^{\cdot}B'$$

by chasing x around the upper right corner of the square, and

$$x + .Z \mapsto p_B(y) + {}^{\cdot}B \mapsto f(p_B(y)) + {}^{\cdot}B' = p_B(f(y)) + {}^{\cdot}B' = p_B(y') + {}^{\cdot}B'$$

around the lower left corner. Thus, the isomorphism is a natural one.

□

So in that precise sense, $\det(.Z)$ and $\det({}^{\cdot}B)$ are the same functor. We write this functor as Crit, and as usual we write $.\mathrm{Crit}$ or ${}^{\cdot}\mathrm{Crit}$ depending on which variance we mean.

The construction of a ROD from an arrow $\partial : C_1 \to C_0$, as in 2 is functorial. Consider the category pre-$\mathcal{A}dj(\mathbb{Z})$ whose objects are adjoint pairs of arrows $\partial : \mathbb{Z}C_1 \to \mathbb{Z}C_0$ and $\partial^* : \mathbb{Z}C_0 \to \mathbb{Z}C_1$ between free $\mathbb{Z}$-modules with an orthonormal inner product, and whose arrows are pairs of arrows $f = (f_1 : \mathbb{Z}C_1 \to \mathbb{Z}C_1', f_0 : \mathbb{Z}C_0 \to \mathbb{Z}C_0'$. If we tensor with $\mathbb{R}$ to obtain $\mathbb{R}C_i$, etc, then $(\mathbb{R}C_1, \mathbb{Z}C_1, \ker(\partial), \mathrm{im}(\partial^*))$ is a ROD, and this assignment is functorial to pre-$\mathcal{ROD}$. A map in pre-$\mathcal{A}dj$ is type (i) if f commutes with ∂, and type (ii) if f commutes with ∂^*. $\mathcal{A}dj$ is the subcategory of arrows that are type (i) and type (ii), and this maps to $\mathcal{ROD}$ under the functor.

We similarly define (pre-)$\mathcal{A}dj(R)$ for an arbitrary (commutative, unital) ring R, where we consider free modules equipped with forms and orthonormal bases.

Given an adjoint pair of arrows, we form $\partial\partial^* : C_0 \to C_0$. We call the sequence $C_0 \to C_0 \to \mathrm{coker}(\partial\partial^*) \to 0$ the "Kirchoff Presentation." We have $\mathrm{im}(\partial\partial^*) = \partial\partial^*(C_0) = \partial({}^{\cdot}B) = \partial({}^{\cdot}B + .Z)$. Thus the map $C_1 \to C_0 \to \mathrm{coker}\partial\partial^*$ kills ${}^{\cdot}B + .Z$, and so we have an induced injection $C_1/(.Z + {}^{\cdot}B) = \mathrm{Crit} \to \mathrm{coker}(\partial\partial^*)$. In fact we have $\mathrm{coker}(\partial\partial^*) = \mathrm{Crit} \oplus \mathrm{coker}(\partial)$.

Note that if f is a map in pre-$\mathcal{A}dj$, the square

$$\begin{array}{ccc} C_0 & \xrightarrow{\partial\partial^*} & C_0 \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_0}\downarrow \\ C_0' & \xrightarrow{\partial\partial^*} & C_0' \end{array}$$

commutes if f is both type (i) and (ii); that is if it is in $\mathcal{A}dj$. In this case we get an induced map on the cokernels. This map restricts to a map of the Critical summand, and agrees with the induced map on Critical groups defined earlier.

3. The order of the critical group

Let C be a ROD. Let $\tau = o(\mathrm{Crit}(C)) = o(.\mathrm{Crit}(C)) = o({}^{\cdot}\mathrm{Crit}(C))$.

Let $\Lambda \subset V$ be a lattice. Let $e_1, ..., e_k$ be an orthonormal basis for V and let $l_1, \ldots, l_k$ be a basis for Λ. Then

$$o(\det(\Lambda)) = \det(\langle l_i, e_j \rangle)^2$$

and this is the volume squared of the parallelogram spanned by the l_i. Such a parallelogram is called a *fundamental region*.

If $v_1, \ldots, v_k$ are k vectors in n-dimensional Euclidean space, then the k-volume of the parallelogram spanned by the vectors is the determinant of the Gram matrix $G = \langle v_i, v_j \rangle$. If we pick an orthonormal basis for the space, say $e_1, \ldots, e_n$, then we have by the Binet-Cauchy theorem

$$\det(G) = \sum_S \det(\langle v_i, e_j \rangle | j \in S)^2$$

where the sum is over all k-element subsets of $\{e_1, \ldots, e_n\}$.

Let $z_1, \ldots, z_m$ be a lattice basis for $.Z$ and $b_1, \ldots, b_\mu$ be a lattice basis for $\dot{}B$. We have $\tau = \det(G(z_i)) = \det(G(b_i))$. That is

$$\tau = \sum_S \det(\langle z_i, e_j\rangle | j \in S) = \sum_T \det(\langle b_i, e_j\rangle | j \in T)$$

where S is an m- and T is a μ-element subset of $\{e_1, \ldots, e_n\}$. There is of course a correspondence between S's and T's given by set complement.

Proposition 10. Corresponding summands are equal. That is, if $S^c = T$ then

$$\det(\langle z_i, e_j\rangle | j \in S) = \det(\langle b_i, e_j\rangle | j \in T)$$

Proof. We assume that the determinant is not zero. The case when it is zero is treated below. Without loss of generality we can take $S = \{e_1, \ldots, e_m\}$ and $T = \{e_{m+1}, \ldots, e_{m+\mu}\}$. (Recall $m + \mu = n$) Let Z be the $n \times m$ matrix $\langle e_i, z_j\rangle$. We have $Z = P \cdot [I_m|X]$ for an $m \times m$ invertible matrix P and an $\mu \times m$ matrix X. Similarly let B be the $n \times \mu$ matrix $\langle e_i, b_j\rangle$, and we have $B = Q \cdot [Y|I_\mu]$. We want to show $(\det P)^2 = (\det Q)^2$.

Let M be the matrix

$$\begin{bmatrix} Z \\ B \end{bmatrix}$$

Now $MM^t =$

$$\begin{bmatrix} ZZ^t & 0 \\ 0 & BB^t \end{bmatrix}$$

The zeroes give us $P(Y^t + X)Q^t = Q(Y + X^t)P^t = 0$ and in particular $Y = -X^t$. The diagonal gives us

$$ZZ^t = P(I_m + XX^t)P^t$$
$$BB^t = Q(I_\mu + YY^t)Q^t = Q(I_\mu + X^tX)Q^t$$

It is a basic result of matrix theory that $\det(I_m + XX^t) = \det(I_\mu + X^tX)$. See [6, Chapter 9 A.1.a] We know $\det ZZ^t = \det BB^t$, and so we conclude $\det PP^t = \det QQ^t$. The proposition follows. □

Then we define the numbers $\rho = \rho(S) = \rho(T)$. ρ is a nonnegative square integer.

If $S \subset \{e_1, \ldots, e_n\}$ and v is a vector then let $S(v)$ be the orthogonal projection of v onto $\mathrm{Span}(S)$. If T is the complement of S then we have of course $v = S(v) + T(v)$. If $\mathrm{Span}(S) \cap .Z = 0$ then call S 'acyclic'.

Proposition 11. $\rho(S) = 0 \iff S$ is *not* acyclic.

Proof. Let T be the complement of S.

$$\begin{aligned} \rho(S) = 0 &\iff \rho(T) = 0 \\ &\iff \text{the vectors } T(l^1), \ldots, T(l^\mu) \text{are linearly dependent} \\ &\iff \text{there are nonzero numbers} a_1, \ldots, a_\mu \text{with} \sum a_i T(l^i) = 0 \end{aligned}$$

Now the l^i are linearly independent, so

$$\sum a_i l^i = \sum a_i T(l^i) + \sum a_i S(l^i) = \sum a_i S(l^i)$$

is not zero. But $\sum a_i S(l^i) \in \mathrm{Span}(S)$, so $\rho(S) = 0 \iff$ there is a nonzero vector in $\mathrm{Span}(.S) \cap .Z$. □

Suppose a ROD C comes from a map $\partial : C \to C_0$, as in 2. Let $e_1, \ldots, e_n$ be a basis for C and $v_1, \ldots, v_{\sigma_0}$ be a basis for C_0. Then writing ∂ as a matrix, the columns of ∂ correspond to $\partial^*(v)$, which of course span $\dot{} B$. The following proposition will be useful in the next section:

Proposition 12. Suppose the columns of ∂ contain a basis for $\mathbb{Z}\dot{}B$. Then every ρ is the determinant squared of a $\mu \times \mu$ square submatrix of ∂.

Proof. There is a μ element subset, say A, of $v_1, \ldots, v_{\sigma_0}$ that corresponds to a basis for $\dot{}B$. If T is a μ element subset of $\{e_1, \ldots, e_n\}$, then $\rho(T)$ is the determinant squared of the submatrix with columns taken out of A and rows taken out of T. □

4. Graphs and Spanning Trees

We take our graphs to be CW complexes $\Gamma = (V, E)$, so that for each edge $e \in E$ there is a map $I \to \Gamma$ so that the endpoints go to vertices and the interior of I is mapped homeomorphically onto it's image. (Here I is the closed interval). If two such maps differ by an orientation-preserving homeomorphism on the right, we regard them as the same. If e denotes the map from I, then $e(1)$ and $e(0)$ are well-defined. Thus, there are 2 such maps for each edge. A choice of one of them is an orientation of that edge. There are also degenerate edges which can be considered constant maps $I \to V$. The set of all oriented edges (including degenerate ones) is denoted $\mathbb{E}$. There is an action of S_2, the symmetric group on 2 letters on this set. If $e \in \mathbb{E}$ then we write $-e$ for the oppositely oriented edge ($-e = e$ if e is degenerate). Define $\sigma_0 = |V|$, $\sigma_1 = |E|$. Then $|\mathbb{E}| = 2\sigma_1 + \sigma_0$.

If v is a vertex, the "in-neighborhood" of v is $\text{in}(v) = \{e \in \mathbb{E} | e(1) = v\}$.

Recall the chain groups. For R a (commutative, unital) ring we write RC_0 for the free module on V and RC_1 for the free module on $\mathbb{E}$ modulo the relations $e + (-e) = 0$ (so if e is degenerate, $e = 0$ in C_1). We define a symmetric nonsingular bilinear form $RC_i \otimes RC_i \to R$ for each i by linearly extending the Kronecker delta. ($\langle e, e' \rangle = 1$ if $e = e'$ and -1 if $e = -e'$. Then we define $\partial : RC_1 \to RC_0$ by $\partial(e) = e(1) - e(0)$ where $e(i)$ is the evaluation of the cell map at i. Taking R to be $\mathbb{Z}$ or $\mathbb{R}$ we can construct a ROD as in section 1.

We have $\dim(C_i) = \sigma_i$. The set V is a basis for C_0 and $\mathbb{E}$ spans C_1. Picking a basis out of $\mathbb{E}$ is the same thing as picking an orientation of Γ. Both of these bases are orthonormal.

In the case of graphs, $n = \sigma_1$ and $\mu = \sigma_0 - 1$. We have the equation $\sum_{v \in V} \partial^* v = 0$, so the set $\{\partial^* v\}$ contains a μ- element basis (in fact, any μ-element subset is a basis). Thus the numbers ρ are determinants squared of submatrices of ∂. We have

Proposition 13. The determinant of any square submatrix of ∂ is 0, +1, or −1.

which is proved in [3]. As a corollary, any ρ which is not zero is one, so $\tau =$ the number of acyclic μ-element subsets of $e_1, \ldots, e_n$. Such a set is precisely a spanning tree of Γ, so

Proposition 14. For a graph Γ, the order of $\text{Crit}(\Gamma)$ is the number of spanning trees in Γ.

Consider the Kirchoff presentation. The map $\partial\partial^* C_0 \to C_0$ is called the Kirchoff matrix, whence the name. $\text{coker}\partial = H_0(\Gamma) = \mathbb{Z}^k$, where k is the number of connected components of Γ. Thus, the critical summand of the cokernel is found by striking out k rows and columns of the Kirchoff matrix. This together with 14 is the usual "Matrix-Tree Theorem."

5. Maps of Graphs

Suppose Γ, Γ' are graphs. A simplicial map $f : \Gamma' \to \Gamma$ is a pair of functions $f_1 : E' \to E$, $f_0 : V' \to V$ that are compatible in the sense that the induced function $f : \mathbb{E}' \to \mathbb{E}$ has $f(-e) = -f(e)$. The induced maps on chain groups satisfy $\partial \circ f = f_0 \circ \partial$, so a simplicial map induces a type (i) map $C_1' \to C_1$.

The following definition is given in [2, 5.7], almost verbatim but translated into our notation:

Definition 15. Let Γ be a connected graph with vertices $v_1, \dots, v_{\sigma_0}$, and let l_{ij} denote the number of edges linking vertices v_i and v_j. We will say that a graph H is *divisible* by Γ if the vertices of Γ' can be partioned into σ_0 classes $U_1, \dots, U_{\sigma_0}$, such that for i,j, a vertex v in U_i is either joined only to vertices in U_i or for every $j \neq i$ is joined to exactly l_{ij} vertices of U_j (and any number of vertices in U_i).

In case Γ' is divisible by Γ, there is a simplicial map $p : \Gamma' \to \Gamma$ that takes a vertex $v \in U_i$ to v_i. This map of course depends on the partition classes and so may not be unique. We therefore make the following definition

Definition 16. $p : \Gamma' \to \Gamma$ is a *Berman bundle* if Γ' is divisible by Γ and p is an associated projection map.

If $p : \Gamma' \to \Gamma$ is any map, define the fiber over a vertex $v \in \Gamma$ to be

$$F_v = \{e' \in \Gamma' | p(e') = v\}$$

We have the following characterization of Berman bundles:

Proposition 17. $p : \Gamma' \to \Gamma$ is a Berman bundle if for every $v' \in \Gamma'$, $\mathrm{in}(v') \subset F_{p(v')}$ or $p : \mathrm{in}(v') - F_{p(v')} \to \mathrm{in}(p(v'))$ is a bijection.

We can phrase this as "if after 'emptying out' the fibers, we are left with a covering space, then p is a Berman bundle." Thus for example covering spaces are Berman bundles.

Proposition 18. Given f,f_0, we have

- (i) f is type (i) whenever $\langle \partial \circ f(e'), v\rangle = \langle f_0 \circ \partial(e), v'\rangle$ for every e', v.
- (ii) f is type (ii) whenever $\langle e, f \circ \partial^*(v')\rangle = \langle \partial(e), f_0(v')\rangle$

Proof. f is type (i) if $\partial \circ f = f_0 \circ \partial$ and f is type (ii) if $\partial \circ f^* = f_0^* \circ \partial$. To conclude f is of a certain type, we only have to check that these equalities hold for the spanning set $\mathbb{E}$ or $\mathbb{E}'$. Finally two vectors x and y are equal if and only if $\langle x, e\rangle = \langle y, e\rangle$ for every e in some orthonormal basis. □

In view of the proposition, we make the calculations

$$\begin{aligned} \langle e, \partial^* \circ f_0(v')\rangle &= 1 \text{ if } e \in \mathrm{in}(f_0(v')), \\ &= -1 \text{ if } -e \in \mathrm{in}(f_0(v')), \\ &= 0 \text{ otherwise.} \end{aligned}$$

$$\langle e, f_0 \circ \partial^*(v')\rangle = \sum_{e' \in \mathrm{in}(v')} \langle e, f(e')\rangle$$

Suppose f is the projection of a Berman bundle. Then for every v',

$$\begin{aligned} \langle e, f \circ \partial^*(v')\rangle &= 1 \text{ if } e \in \mathrm{in}(f(v')) \\ &= -1 \text{ if } -e \in \mathrm{in}(f(v')) \\ &= 0 \text{ otherwise.} \end{aligned}$$

so f induces a type (ii) map.

We therefore have

Proposition 19. If $p : \Gamma' \to \Gamma$ is the projection of a Berman bundle, then p induces a map $.\text{Crit}(\Gamma') \to .\text{Crit}(\Gamma)$. Since p is surjective, so is the the induced map.

Corollary 20. [2, Theorem 5.7] If Γ divides Γ', then $\tau(\Gamma)$ divides $\tau(\Gamma')$.

Let $\Gamma = (V, E)$ be a graph, and construct $\mathbb{E}$ as usual. Let $E_{nd} \subset E$ be the set of nondegenerate edges of Γ, and let $\mathbb{E}_{nd}$ be the set of nondegenerate oriented edges.

If $\Gamma = (V, E)$, $\Gamma' = (V', E')$ are two graphs, consider a pair of maps $l_0 : V \to V'$ and $l_1 : E'_{nd} \to E_{nd}$. We will call this a pre-Lorenzini map $l = (l_0, l_1) : \Gamma' \to \Gamma$. (The fact that we define this to be a map $\Gamma \to \Gamma'$ and not $\Gamma' \to \Gamma$ is completely arbitrary, of course). We have induced maps $l_0 : C_0 \to C'_0$ and $l_1 : C'_1 \to C_1$, and the following squares:

$$\begin{array}{ccccccc} C'_1 & \xrightarrow{\partial} & C'_0 & & C'_1 & \xleftarrow{\partial^*} & C'_0 \\ {\scriptstyle l_1}\downarrow & & \downarrow{\scriptstyle l_0^*} & & {\scriptstyle l_1}\downarrow & & \downarrow{\scriptstyle l_0^*} \\ C_1 & \xrightarrow{\partial} & C_0 & & C_1 & \xleftarrow{\partial^*} & C_0 \end{array}$$

If the left square commutes, then we say that l is type (i), and if the right square commutes we say that it is type (ii). If a pre-Lorenzini map is both type (i) and type (ii), then we call it a Lorenzini map. Lorenzini maps therefore induce maps on the tree groups.

If Γ' is obtained from Γ by subdividing each edge of Γ then we have an inclusion map $l_0 : V \to V'$ and a map $l_1 : E' \to E$ that takes each edge to the edge that it was "originally a part of."

Proposition 21. [5] For any k, the map l just defined is a Lorenzini map if each edge is subdivided into k parts. The induced map $.\text{Crit}(\Gamma') \to .\text{Crit}(\Gamma)$ is onto, and the kernel is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$. That is, we have the following short exact sequence:

$$0 \to (\mathbb{Z}/k\mathbb{Z})^m \to \text{Crit}(\Gamma') \to \text{Crit}(\Gamma) \to 0$$

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