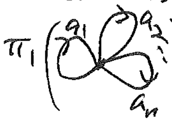
Comb. Prob. Session Feb 27, 2015

For $G := \mathrm{Out}(F_n)$ = outer automorphisms of free group $F_n = \langle a_1, \dots, a_n \rangle$ on n elements

$= \pi_1\Big($  $\Big)$

$$:= \mathrm{Aut}(F_n) \Big/ \underbrace{\mathrm{Inn}(F_n)}_{\text{inner automorphisms}}$$

inner automorphisms $= \{\text{conjugations by } g \text{ in } G\}$

$x \longmapsto gxg^{-1}$

$\big($so $1 \longrightarrow \mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(F_n) \longrightarrow \mathrm{Out}(F_n) \to 1\big)$

Culler & Vogtmann 1986 produced such an $X \circlearrowleft G$ stabilizers finite

the spine of outer space!

$\downarrow$

$X/G = \Sigma$ finite

Smillie & Vogtmann 1987 used it ~~to~~ show
J Pure Appl. Alg

$$\chi(\mathrm{Out}(F_n)) = \chi_{1,n} + \chi_{2,n} + \dots + \chi_{2n-2,\,3n-3}$$

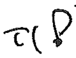
CONJ: $\chi(\mathrm{Out}(F_n)) < 0$ and grows exponentially in n

$$\chi_{v,e} := \frac{1}{v!\,e!\,2^e} \sum_{\substack{G \text{ admissible} \\ V = \{1,2,\dots,v\} \\ E = \{1,2,\dots,e\}}}' \tau(G)$$

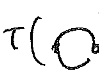
where $\tau(G) = (-1)^{v-1} T_G(0,1)$

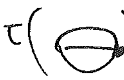
$$= \sum_{\substack{\text{subforests} \\ F \subset G}} (-1)^{|F|}$$

$\tau(\bullet) = +1$

$\tau($  $) = +1$

$\tau($ $) = 0$

$\tau($  $) = -1$

$\tau($  $) = -2$

G a group $\rightsquigarrow$ group cohomology $H^\bullet(G) \overset{\text{DEF'N}}{=} H^\bullet(BG)$

where $* \approx EG \circlearrowleft G$ free, i.e. all cells have <u>trivial</u> stabilizers

$$\begin{array}{c} * \approx EG \circlearrowleft G \\ \downarrow \\ BG = EG/G \end{array}$$

If BG was a <u>finite</u> cell complex,

then
$$\chi(G) \overset{\text{DEF'N}}{:=} \sum_{i \geq 0} (-1)^i \operatorname{rank} H^i(BG)$$
$$= \sum_{\text{cells } \sigma \text{ in } BG} (-1)^{\dim(\sigma)}$$

Euler-Poincaré

Sometimes they can't find such an $\overset{*}{\underset{\approx}{}} EG \underset{G}{\circlearrowleft}$ free and $BG = EG/G$ with BG finite,

but instead they can find $* \approx X \circlearrowleft G$ with stabilizers of cells <u>finite</u>,

$$\begin{array}{c} X \\ \downarrow \\ X/G = \boxtimes \text{ finite} \end{array}$$

and compute the <u>rational</u> <u>Euler</u> <u>characteristic</u>

$$\chi(G) := \sum_{\sigma \in \boxtimes} \frac{(-1)^{\dim(\sigma)}}{|\operatorname{stab}_G(\sigma)|}$$

In[13]:= Series[Chi,{z,0,6}]

$$\text{Out[13]}= -\frac{z}{24} - \frac{z^2}{48} - \frac{161\,z^3}{5760} - \frac{367\,z^4}{5760} - \frac{120257\,z^5}{580608} - \frac{39793\,z^6}{45360} + O[z]^7$$

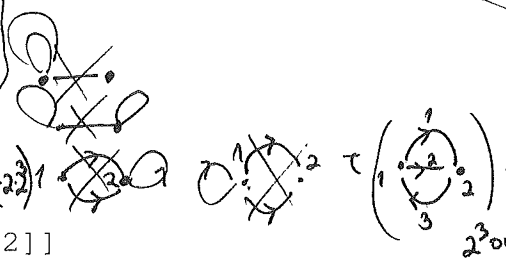
In[5]:= Cantidiag[2]

$\chi_{1,2}$

$\tau(\infty) = +1$ $\quad v=1, e=2 \quad v=2, e=3$ sums to $\left(-\frac{1}{24}\right)$

$$\text{Out[5]}= \left\{\frac{1}{8}, -\left(\frac{1}{6}\right)\right\}$$

$\frac{1}{1!\,2!\,2^2}(+1) \qquad \frac{1}{2!\,3!\,2^3}(-2^3)$



$\tau(\;) = 2$ $\quad 2^3$ orientations

In[6]:= N[Cantidiag[2]]

Out[6]= {0.125, -0.166667}

In[7]:= Cantidiag[3]

$$\text{Out[7]}= \left\{\frac{1}{48}, -\left(\frac{7}{24}\right), \frac{3}{4}, -\left(\frac{1}{2}\right)\right\}$$ ⟵ sums to $\left(-\frac{1}{48}\right)$

In[8]:= N[Cantidiag[3]]

Out[8]= {0.0208333, -0.291667, 0.75, -0.5}

In[9]:= Cantidiag[4]

$$\text{Out[9]}= \left\{\frac{1}{384}, -\left(\frac{31}{160}\right), \frac{583}{288}, -\left(\frac{163}{24}\right), \frac{71}{8}, -\left(\frac{71}{18}\right)\right\}$$

In[10]:= N[Cantidiag[4]]

Out[10]= {0.00260417, -0.19375, 2.02431, -6.79167, 8.875, -3.94444}

In[11]:= Cantidiag[5]

$$\text{Out[11]}= \left\{\frac{1}{3840}, -\left(\frac{113}{1280}\right), \frac{175}{64}, -\left(\frac{14473}{576}\right), \frac{1559}{16}, -\left(\frac{8665}{48}\right), \frac{633}{4}, -\left(\frac{211}{4}\right)\right\}$$