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Combinatorics Problem Session 4/17/2015

In particular, K is annihilated, and hence preserved, by every one of the self-adjoint operators $\{\nu_{(2^k,1^{n-2k})}\}_{k=0,1,2,\ldots,k_{\max}}$.

Hence they also preserve the perpendicular space $U := K^\perp$, a $\mathbb{Q}$-rational subspace of $\mathbb{R}\mathfrak{S}_n$. Note that Corollary IV.2.2 implies that the $\mathfrak{S}_n$-representation afforded by U is the same as that afforded by the non-zero eigenspaces of a certain BHR operator b_J. Meanwhile, Example IV.6.3 shows that this $\mathfrak{S}_n$-representation is the sum of $\bigoplus_{a=0}^{k_{\max}} \mathrm{WH}_{\mathcal{O}_{(2^a,1^{n-2a})}}$. Note that this sum is isomorphic to the multiplicity-free Gelfand model described in Proposition V.1.1.

This multiplicity-freeness has two consequences. First, it shows that by combining the $\mathfrak{S}_n$-isotypic decomposition $U = \bigoplus_\lambda U^\lambda$ together with the complementary space K, one obtains a direct sum decomposition as in (20) that simultaneously diagonalizes all of the operators $\{\nu_{(2^k,1^{n-2k})}\}_{k=0,1,2,\ldots,k_{\max}}$.

Secondly, the eigenvalue integrality principle, Proposition I.3.1, implies that each operator $\nu_{(2^k,1^{n-2k})}$ acts on U with integer eigenvalues. Since $\nu_{(2^k,1^{n-2k})}$ also annihilates the subspace $K = U^\perp$ complementary to U, it has only integer eigenvalues on all of $\mathbb{R}\mathfrak{S}_n$.

However, we know more about the eigenvalue[1] $\gamma_{(2^k,1^{n-2k}),\lambda}$ with which $\nu_{(2^k,1^{n-2k})}$ acts on U^λ. Picking any realization $\mathfrak{S}_n \xrightarrow{\rho_\lambda} GL_{\mathbb{C}}(V)$ of the irreducible $\mathfrak{S}_n$-representation with character χ^λ, Proposition II.7.1 tells us that $\rho_\lambda(\nu_{(2^k,1^{n-2k})})$ has $\gamma_{(2^k,1^{n-2k}),\lambda}$ as its only potential non-zero eigenvalue, and hence

$$\boxed{\begin{aligned}\gamma_{(2^k,1^{n-2k}),\lambda} &= \mathrm{Trace}\,\rho_\lambda(\nu_{(2^k,1^{n-2k})})\\ &= \mathrm{Trace}\left(\sum_{w\in\mathfrak{S}_n} \mathrm{noninv}_{(2^k,1^{n-2k})}(w)\cdot\rho_\lambda(w)\right)\\ &= \sum_{w\in\mathfrak{S}_n} \mathrm{noninv}_{(2^k,1^{n-2k})}(w)\cdot \mathrm{Trace}\,\rho_\lambda(w)\\ &= \sum_{w\in\mathfrak{S}_n} \mathrm{noninv}_{(2^k,1^{n-2k})}(w)\cdot\chi^\lambda(w).\end{aligned}}$$

Lastly, to see how $\mathbb{Z}_2$ acts on U^λ, note that Proposition V.1.1 implies that U^λ lies in

$$\mathrm{im}(\nu_{(2^a,1^{n-2a})}) \cap \mathrm{im}(\nu_{(2^{a-1},1^{n-2a+2})})^\perp \tag{23}$$

where $a := \frac{n-\mathrm{oddcols}(\lambda)}{2}$. Since $\nu_{(2^a,1^{n-2a})}, \pi_{(2^a,1^{n-2a})}$ share the same kernels, one has an isomorphism of $\mathbb{R}[\mathfrak{S}_n\times\mathbb{Z}_2]$-modules

$$\mathrm{im}(\nu_{(2^a,1^{n-2a})}) \cong \mathrm{im}(\pi_{(2^a,1^{n-2a})}).$$

Consequently the space (23) carries $\mathbb{R}[\mathfrak{S}_n\times\mathbb{Z}_2]$-module structure isomorphic to that of

$$\mathrm{im}(\pi_{(2^a,1^{n-2a})})/\mathrm{im}(\pi_{(2^{a-1},1^{n-2a+2})})$$

which is $\mathrm{WH}_{\mathcal{O}_{(2^a,1^{n-2a})}} \otimes (\chi^-)^{\otimes a}$ by Example IV.6.3. Thus, $\mathbb{Z}_2$ acts by $(\chi^-)^{\otimes a}$ on U^λ. □

REMARK 2.2. One does not have that the associated BHR-operators b_J pairwise commute, in contrast with the situation for the original family $\{\nu_{(k,1^{n-k})}\}_{k=1,2\ldots,n}$.

[1] The first author thanks C.E. Csar for discussions leading to this expression for $\gamma_{(2^k,1^{n-2k}),\lambda}$.

From R.-Saliola-Welker
"Spectra of symmetrized shuffling operators"
Memoirs of AMS #1072,
(2014).

REMARK 2.3. The formula for the eigenvalue $\gamma_{(2^k,1^{n-2k}),\lambda}$ given in (21) is somewhat explicit, but still leaves something to be desired. For example, the character values $\chi^\lambda(w)$ for w in $\mathfrak{S}_n$ are integers, but they can be negative. Thus (21) does not manifestly show the fact that $\gamma_{(2^k,1^{n-2k}),\lambda}$ is *non-negative*, nor does it show the fact that $\gamma_{(2^k,1^{n-2k}),\lambda}$ vanishes unless $\mathrm{oddcols}(\lambda) \geq n-2k$. This suggests the following problem.

PROBLEM 2.4. For each partition λ of n, and each k with $\mathrm{oddcols}(\lambda) \geq n-2k$, find a more explicit formula for the non-zero eigenvalue $\gamma_{(2^k,1^{n-2k}),\lambda}$ of $\nu_{(2^k,1^{n-2k})}$ acting on its (non-kernel) eigenspace U^λ affording χ^λ.

We have computed some of these eigenvalues using Sage [68], and we present this data for $3 \leq n \leq 6$ in the tables below. The data is presented as follows:

- each row of the table corresponds to the subspace U^λ affording χ^λ;
- the entry in the column indexed by $\nu_{(2^k,1^{n-2k})}$ is the eigenvalue $\gamma_{(2^k,1^{n-2k}),\lambda}$;
- the entry in the column indexed by w_0 is the eigenvalue for the $\mathbb{Z}_2$-action on U^λ.

To enhance the presentation of the data, every zero eigenvalue has been replaced by a dot.

DEF'N:
If $w = (w_1, \ldots, w_n) \in \mathfrak{S}_n$, then

$\mathrm{noninv}_{(2^k,1^{n-2k})}(w) = \#\{$ partial matchings $\{\{i_1,j_1\},\{i_2,j_2\},\ldots,\{i_k,j_k\}\}$ of $\{1,2,\ldots,n\}$ with k arcs : $w_{i_1} < w_{j_1},\ w_{i_2} < w_{j_2}, \ldots, w_{i_k} < w_{j_k}\}$

with $i_1 < j_1,\ i_2 < j_2, \ldots, i_k < j_k$