COMB. PROBLEM SESSION Sept. 5, 2014 V. Reiner

Q: Why do these symmetric matrices $\{\nu_k\}_{k=1,\dots,n}$ pairwise commute?

$$\mathbb{Z}^{n! \times n!} = \mathbb{Z}^{S_n \times S_n}, \quad S_n = \text{symmetric group on } \{1,2,\dots,n\}$$

$$(\nu_k)_{v,u} := \operatorname{noninv}_k(v^{-1}u)$$

where $\operatorname{noninv}_k(w) := \#\{1 \le i_1 < \dots < i_k \le n : w_{i_1} < \dots < w_{i_k}\}$

"k-noninversions of w"

$= \#\{\text{inc. subsequences of length } k \text{ in } w\}$

$= \#$ patterns $12\cdots k$ inside w

$= \operatorname{noninv}_k(w^{-1})$

THM (THM I.1.1 in Saliola-Welker-R. Mem AMS 228) $\nu_i \nu_j = \nu_j \nu_i \quad \forall i,j$

proof: An unenlightening induction on n ∎

$\boxed{\text{PROBLEM}}$ Give an insightful proof.

EXAMPLE: $n=3$

$$\nu_n = \nu_3 = \begin{array}{c|c} & 123 \ \cdots \ 321 \\ \hline \begin{array}{c}123\\ \vdots \\ \vdots \\ 321\end{array} & \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{bmatrix} \end{array}$$

$$\nu_1 = \begin{array}{c|c} & 123 \ \ 132 \ \cdots \ 321 \\ \hline \begin{array}{c}123\\ 132 \\ \vdots \\ 321\end{array} & \begin{bmatrix} 3 & 3 & \cdots & 3 \\ 3 & 3 & & \vdots \\ \vdots & & \ddots & \vdots \\ 3 & \cdots & \cdots & 3 \end{bmatrix} \end{array} = n \cdot I_{n!}$$

$$\nu_2 = \begin{array}{c|c} & \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \hline \begin{array}{c} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} & \begin{bmatrix} 3 & 2 & 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 1 \\ 2 & 1 & 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 & 1 & 2 \\ 1 & 0 & 2 & 1 & 3 & 2 \\ 0 & 1 & 1 & 2 & 2 & 3 \end{bmatrix} \end{array}$$

row and column sums constant, so commutes with ν_1.

RMKS: (all in above reference)

① ν_k is really expressing $\mathbb{Z}S_n \xrightarrow{\cdot \nu_k} \mathbb{Z}S_n$

$\mathbb{Z}S_n$ = group algebra $= \{\sum_{\sigma \in S_n} c_\sigma \sigma : c_\sigma \in \mathbb{Z}\}$

$\nu_k = \sum_{\sigma \in S_n} \operatorname{noninv}_k(\sigma) \cdot \sigma$

so its eigenspaces become S_n-representations.

(2) $\nu_k = b_k\, b_k^T$ (up to scaling)

where $\{b_k\}_{k=1,2,\ldots,n}$ are Markov matrices for some BHR random walks on S_n, and $\{b_k\}_{k=1,2,\ldots,n}$ do commute (for insightful reasons)

Bidigare-Hanlon-Rockmore

(when $k=2$, b_2 = random-to-top shuffling
ν_2 = random-to-random)

(3) $\{\nu_k\}$ generalize from $S_n = W$ to finite reflection groups W and operators ν_J for (conj. classes of) parabolic subgroups $W_J = \langle J \rangle$

Q: When do $\nu_J, \nu_{J'}$ commute?

e.g. In type B_n, $\{\nu_k\}_{k=1,\ldots,n}$ seem to have an analogous commuting family

(4) Q: Is there a q-analogue of $\{\nu_k\}$, e.g. in $\underbrace{\mathcal{H}_n(q)}_{\text{Hecke algebra for } S_n}$?