THE FOUCAULT PENDULUM (WITH A TWIST)

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Abstract. A Foucault pendulum is supposed to precess in a direction opposite to the earth's rotation, but nonlinear terms in the equations of motion can also produce precession. The goal of this paper is to study the motion of a nonlinear, spherical pendulum on a rotating planet. It turns out that the problem on a fixed energy level reduces to the study of a monotone twist map of an annulus. For certain values of the parameters, this leads to existence proofs for orbits which do not precess or else precess in the wrong direction. In fact there will be nonprecessing periodic solutions which return to their initial state after swinging back and forth just once. For pendula of modest size, these nonprecessing periodic solutions can be very nearly planar.

The Foucault pendulum is often given as proof of the rotation of the earth. As the pendulum swings back and forth, the positions of maximum amplitude precess in a direction opposite to the earth's rotation. But a nonlinear spherical pendulum on a nonrotating planet also exhibits precession. So the observed precession of Foucault's pendulum must be a combination of two effects. The goal of this paper is to see how the two precessions interact to produce the observed motion. In the usual Foucault experiment, the initial conditions are chosen to be such that the pendulum motion is nearly planar. If we consider more general initial conditions, it turns out that some surprising motions are possible. In particular, the direction of precession can be reversed or stopped altogether.

Mathematically we find that the problem reduces to a study of a monotone twist map of an annulus. The boundary circles of the annulus are nearly circular periodic orbits of the pendulum, that is, instead of swinging back and forth in a nearly planar motion, the pendulum is sweeping out a circular cone, moving either clockwise or counterclockwise. The points near the boundary of the annulus represent nearly circular elliptical orbits which are slowly precessing. For certain values of the parameters, including those suggested by the original Foucault pendulum, the directions of precession of these orbits are opposite. It follows from the Poincaré-Birkhoff theorem that there are periodic motions of the pendulum which do not precess at all. The precession due to the rotation of the earth is completely canceled by

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the precession due to the nonlinearity of the pendulum and the pendulum returns to its initial state after one swing. Depending on the length and amplitude of the pendulum, the shape of these nonprecessing periodic orbits can be highly eccentric ellipses, disturbingly close to planar.

Figure 1 illustrates some of our main results. The precession rate for a pendulum on the earth is rather slow so the figure shows a numerical simulation of a more rapidly rotating planet. If the pendulum is released with zero initial velocity, the motion is as expected. However, a small initial rotation will produce qualitatively different behavior due to the nonlinearity of the pendulum. For a pendulum of moderate size on the earth, these counterintuitive motions occur for smaller initial velocities than shown here.

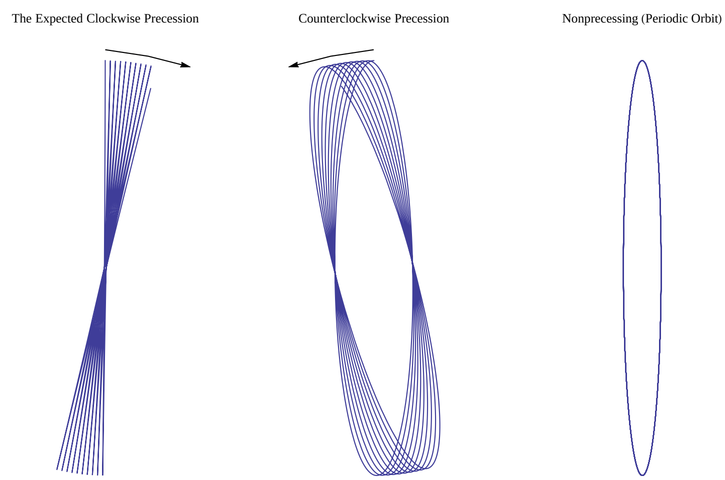

FIGURE 1. Some orbits of a Foucault pendulum on a rapidly rotating planet. The expected behavior in the Northern hemisphere is clockwise precession. However, a small counterclockwise initial velocity can produce precession in the opposite direction or a nonprecessing periodic motion. The precession rate is exaggerated here for clarity.

The qualitative behavior of a Foucault pendulum motion depends on the length of the pendulum, its location on the earth and how the pendulum is set into motion. For motions like those figure 1 to be possible, the amplitude of the pendulum motion has to be large enough for the nonlinear effects to overcome the rotation of the earth. Figure 2 plots a variable ϵ representing the amplitude against a *rotation parameter*, $\rho \sin(\lambda)$. Here ρ is the small parameter giving the ratio of the frequency of the earth's rotation to the natural frequency of the pendulum and λ is the latitude. The ratio $\kappa = \rho \sin\lambda/\epsilon^2$ turns out to be a crucial quantity. The various counterintuitive behaviors we are after all occur if $-\frac{3}{16} < \kappa < \frac{3}{16}$ (shaded region in the figure).

A given Foucault pendulum, such at the one demonstrated by Foucault at the Pantheon in Paris, has a fixed value of the rotation parameter and this determines a horizontal line in figure 2. Choosing an amplitude far enough to the right on this line makes all of the interesting phenomena possible. As the figure shows, the amplitude Foucault used does fall in the shaded region.

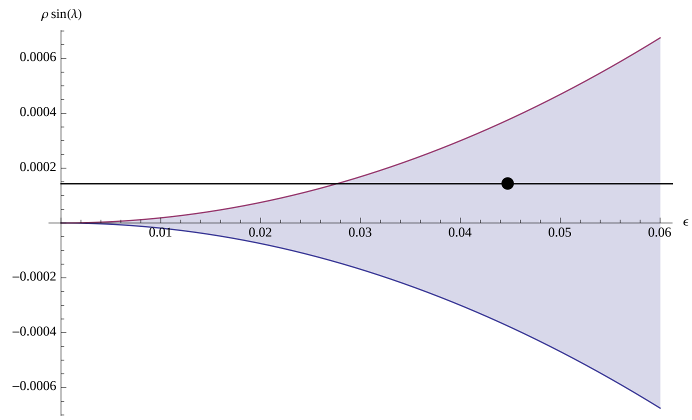

FIGURE 2. Parameter values leading to the existence of counterintuitive behavior. The important parameters are the amplitude (horizontal) and the rotation parameter $\rho\sin(\lambda)$ (vertical). The shaded region shows which parameters admit motions as in figure 1. The line represents the pendulum demonstrated by Foucault in the Pantheon in Paris and the dot indicates the amplitude he chose.

Of course the precession of the spherical pendulum is a well-known phenomenon and after Foucault's famous demonstration, the size of the effect was estimated and shown to be much smaller than the precession due to the earth's rotation for sufficiently planar initial conditions [10, 5]. In particular, this is true for initial conditions with zero initial velocity relative to the earth. Here we consider what might happen for general initial conditions, which leads to the twist map and the existence of the nonprecessing periodic solutions.

Here is an outline of the paper. After deriving the equations of motion in a convenient form we calculate a normal form near the downward pointing equilibrium of the pendulum. The qualitative phenomena mentioned above can be found for the normal form and the main results are devoted to showing that they are still present for the full system. Theorem 1 below shows that the relative equilibrium orbits of the normal form (where the pendulum traces out a circular path) can be continued to the full system. These orbits form the boundaries of the annular surface of section whose existence is established in Theorem 2. Once the problem has been reduced to a twist map of the annulus, Theorem 3 about the existence of nonprecessing orbits follows.

1. Equation of Motion

Here is a derivation of equations of motion for the Foucault pendulum. Consider a spherical pendulum near the surface of a rotating planet. We want to describe the motion of the pendulum bob with respect to a noninertial *laboratory* coordinate system which is fixed with respect to the surface of the planet. In order to get the correct equations of motion we will also need to consider an inertial coordinate system, say one that is fixed in space and located at the center of the planet. Initially we will ignore the spherical pendulum constraint and consider the motion of a point with mass m. Define

$$\begin{aligned} q &= (x, y, z) = \text{coordinates in the lab frame} \\ Q &= (X, Y, Z) = \text{coordinates in the inertial frame.} \end{aligned}$$

Assume that the planet is rotating uniformly around the Z-axis at angular speed ω and let

$$R(t) = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be the corresponding rotation matrix. The relationship between the inertial and noninertial coordinates is given by

$$Q(t) = R(t)R_0(q(t) + b_0)$$

where R_0 is an initial rotation and $R_0 b_0$ describes the initial position of the origin of the lab frame. For a Foucault pendulum on the surface of the earth, we have

$$\omega \approx \frac{2\pi}{24 \cdot 60 \cdot 60} \approx 0.00007272\, radians/sec$$

while b_0 is a vector of length $|b_0| \approx 6371\, km$.

The velocity vectors with respect to the two frames are related by

$$\begin{aligned} \dot{Q}(t) &= R(t)R_0\dot{q}(t) + \dot{R}(t)R_0(q(t) + b_0) \\ &= R(t)R_0\left[\dot{q}(t) + \hat{\alpha}\, q(t) + \hat{\alpha}\, b_0\right] \end{aligned}$$

where

$$\hat{\alpha} = R_0^{-1}R(t)^{-1}\dot{R}(t)R_0 = \omega R_0^{-1}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_0$$

is the constant angular velocity matrix in the lab frame. The angular velocity vector associated to the antisymmetric matrix $\hat{\alpha}$ is $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where

$$\hat{\alpha} = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}.$$

Then for any vector $V \in \mathbb{R}^3$, $\hat{\alpha}\, V = \alpha \times V$ where $\times$ denotes the cross product. We have $|\alpha| = \omega$, the angular speed of rotation of the planet. Using this notation, the kinetic energy of a point of mass m with positions $q(t), Q(t)$ with respect to the two frames is given by

$$T = \frac{m}{2}\,|\dot{Q}|^2 = \frac{m}{2}\,|\dot{q}(t) + \alpha \times q(t) + \alpha \times b_0|^2.$$

If the point is moving over distances which are small compared to the size of the planet we can model gravity by a force field which is constant in the lab frame. The force will be given by $-\nabla U(q)$ where the gravitational potential energy is

$$U(q) = -mk\,\gamma \cdot q.$$

Here γ is unit vector giving the constant direction of the gravitational force in the lab frame and the scalar k gives its strength.

The motion of the mass point in the lab frame will be described by the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

for the Lagrangian

$$L = T - U = \frac{m}{2}\,|\dot{q} + \alpha \times q + \alpha \times b_0|^2 + mk\,\gamma \cdot q.$$

Before imposing the spherical pendulum constraint we will make a few simplifications. Since the m appears in both terms, it will drop out of the Euler-Lagrange equations. So we may as well set $m = 1$. Next, expand the kinetic energy as

$$T = \frac{1}{2}\,|\dot{q} + \alpha \times q|^2 + \frac{1}{2}\,|\alpha \times b_0|^2 + \dot{q} \cdot (\alpha \times b_0) + (\alpha \times q) \cdot (\alpha \times b_0).$$

The second term can be dropped since it is constant. The third term does not affect the Euler-Lagrange equations, so it can also be dropped.

The fourth term can be simplified using vector identities to

$$(\alpha \times q) \cdot (\alpha \times b_0) = ((\alpha \times b_0) \times \alpha) \cdot q = |\alpha|^2 b_0^\perp \cdot q$$

where

$$b_0^\perp = b_0 - \frac{\alpha \cdot b_0}{|\alpha|^2}\alpha$$

is the projection of b_0 onto the plane orthogonal to the angular velocity of the planet.

We arrive at the simplified Lagrangian

$$\begin{aligned} L &= \frac{1}{2}\,|\dot{q} + \alpha \times q|^2 + \omega^2 b_0^\perp \cdot q + k\,\gamma \cdot q \\ &= \frac{1}{2}\,|\dot{q} + \alpha \times q|^2 + g\,\nu \cdot q \end{aligned}$$

where

$$g\,\nu = \omega^2 b_0^\perp + k\,\gamma.$$

The vector $g\,\nu$ represents the net acceleration due to gravity and to the centrifugal force from the rotation of the origin of the lab frame around the planet. Here ν is a unit vector which we view as defining the direction "down" in the lab frame and g is a scalar representing the constant downward acceleration.

Now we impose the spherical pendulum constraint:

$$|q|^2 = x^2 + y^2 + z^2 = l^2$$

where $l > 0$ is the length of the pendulum. We can reduce to the case $l = 1$ by the substitution $q \mapsto l\,q$. Dividing the Lagrangian by l^2 gives a new Lagrangian:

$$L = \frac{1}{2}\,|\dot{q} + \alpha \times q|^2 + \frac{g}{l}\,\nu \cdot q = \frac{1}{2}\,|\dot{q} + \alpha \times q|^2 + \omega_0^2\,\nu \cdot q.$$

where $\omega_0 = \sqrt{g/l}$ is the angular speed of the small oscillation of the pendulum, derived from the linearized differential equations. We can eliminate this parameter as well if we change the time scale so that the period of the linearized pendulum is 2π. This amounts to making the substitution $\dot{q} \mapsto \omega_0 \dot{q}$. If we do this and then divide the Lagrangian by ω_0^2 we get:

$$L = \frac{1}{2}\,|\dot{q} + \beta \times q|^2 + \nu \cdot q$$

where $\beta = \alpha/\omega_0$ is the angular velocity vector of the planet using the new time units. It has length $\rho = |\beta| = \omega/\omega_0$, that is, the ratio of the angular speeds of the planet and the linearized pendulum.

Foucault's pendulum in the Pantheon in Paris had length $l \approx 67\,m$. Since $g \approx 9.8\,m/sec^2$ we find

$$\omega_0 \approx 0.3825 \qquad \rho = |\beta| = \omega/\omega_0 \approx 0.00019.$$

Most other Foucault pendulums are not so long and lead to even smaller values of ρ.

After all these simplifications we are ready to impose the spherical pendulum constraint:

$$|q|^2 = x^2 + y^2 + z^2 = 1.$$

According to the principles of Lagrangian mechanics we only need to restrict the Lagrangian function L to the tangent bundle to the sphere, that is, to impose the velocity constraint

$$q \cdot v = 0 \qquad v = \dot{q}.$$

We could derive differential equations by introducing two local coordinates on the sphere, writing L in these coordinates and then computing the corresponding Euler-Lagrange equations. Initially we will use an alternative method which allows us to work on the whole sphere.

The trick is to homogenize the Lagrangian to obtain a Lagrangian on the tangent bundle of $\mathbb{R}^3 \setminus 0$ which agrees with L on the tangent bundle of the sphere and which leaves the sphere invariant. Writing v instead of $\dot{q}$, the new Lagrangian is

$$L(q, v) = \frac{1}{2}\frac{|v + \alpha \times q|^2}{|q|^2} + \frac{\nu \cdot q}{|q|}.$$

This agrees with the previous Lagrangian when $|q| = 1$ but is constructed to be invariant under the scaling symmetry $(q, v) \mapsto (kq, kv)$ where $k > 0$. By Noether's theorem this symmetry gives rise to a conserved quantity

$$\frac{\partial L}{\partial v} \cdot \frac{\partial (kq)}{\partial k} = \frac{(v + \beta \times q) \cdot q}{|q|^2} = \frac{q \cdot v}{|q|^2}.$$

If we restrict to the level $q \cdot v = 0$ then $|q|$ is also constant so all of the spheres $|q| = c$ will be preserved by the Euler-Lagrange flow.

In the rest of the paper we will work with the Hamiltonian version of the system. Let

$$p = \frac{\partial L}{\partial v} = \frac{(v + \beta \times q)}{|q|^2} \qquad v = |q|^2 p - \beta \times q$$

and define

$$H(q, p) = p \cdot v - L = \frac{1}{2}|q|^2|p|^2 - p \cdot (\beta \times q) - \frac{\nu \cdot q}{|q|}.$$

Hamilton's differential equation leave the constraint set set $|q| = 1, p \cdot q = 0$ invariant. On this set they simplify to:

$$\begin{aligned} \dot{q} &= H_p = p - \beta \times q \\ \dot{p} &= -H_q = -|p|^2 q + p \times \beta + \nu - (\nu \cdot q) q. \end{aligned} \tag{1}$$

2. Equilibrium points and the normal form

Intuitively, there should be a stable equilibrium position with the pendulum hanging down. Unfortunately, this is not quite true. To investigate, choose the initial rotation R_0 of the lab coordinates $q = (x, y, z)$ so that the z-axis is vertical (with the positive direction up) and such that the rotation axis of the earth is in the (x, z)-plane. Then we have

$$\nu = (0, 0, -1) \qquad \beta = \rho(\cos\lambda, 0, \sin\lambda) = \rho\,\zeta$$

where $\rho = |\beta| = \omega/\omega_0$ and α is a unit vector aligned with the planet's rotation axis. The angle λ measures the "latitude" of the lab.

With $p = (p_x, p_y, p_z)$, the Hamiltonian becomes

$$\begin{aligned} H(q, p) = \frac{1}{2}(x^2 + y^2 + z^2)(p_x^2 + p_y^2 + p_z^2) + \rho\cos\lambda(z p_y - y p_z) \\ + \rho\sin\lambda(y p_x - x p_y) + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

We will look for an equilibrium point (q_0, p_0) of (1) with $q_0 \approx (0, 0, -1)$. The equations for an equilibrium can be written

$$\begin{aligned} p &= \rho\,\zeta \times q = \rho(-y\sin\lambda, x\sin\lambda - z\cos\lambda, y\cos\lambda) \\ 0 &= (xz, yz, z^2 - 1) + \rho^2\,(\zeta \cdot q)\,((\zeta \cdot q) q - \zeta)\,. \end{aligned}$$

When $\rho = 0$ the second equation gives $q_0 = (0, 0, -1)$ and substituting this into the first equation gives $p_0 = (0, \rho\cos\lambda, 0)$. When $\rho \neq 0$ taking q_0 in the direction of gravity no longer gives an exact equilibrium. Instead we find a family of equilibria with

$$q_0(\rho) = (0,0,-1) + O(\rho^2) \qquad p_0(\rho) = (0, \rho\cos\lambda, 0) + O(\rho^2).$$

This complication arises from the small centrifugal force due to the lab rotating about its own origin. Since there is no simple formula for the exact location of the equilibrium point, we will treat terms of order $O(\rho^2)$ as perturbations. This is reasonable from a physical point of view since other terms of comparable size are already being ignored, for example, terms due to the fact that the gravitational force is not really constant.

We will study the dynamics near the equilibrium using the Taylor series of the Hamiltonian. Up to terms of order $O(\rho^2)$, the Taylor series at $(q_0(\rho), p_0(\rho))$ agrees with the Taylor series at the simpler point $q_0 = (0,0,-1), p_0 = (0, \rho\cos\lambda, 0)$ so we will compute the series there. We will use (x,y) as local coordinates on the sphere near q_0 by setting $z = -\sqrt{1-x^2-y^2}$. Introduce new momentum variables (u,v) and define a symplectic extension of these local coordinates centered at $(0, \rho\cos\lambda, 0)$ by setting

$$p_x = u - x(xu+yv) \quad p_y = \rho\cos\lambda + v - y(xu+yv) \quad p_z = (xu+yv)\sqrt{1-x^2-y^2}.$$

Up to terms of order $O(\rho^2)$ we find

$$\begin{aligned} H = -1 &+ \frac{1}{2}(1-x^2)u^2 - xyuv + \frac{1}{2}(1-y^2)v^2 \\ &+ \rho\cos\lambda(fv - xyu - y^2v) + \rho\sin\lambda(yu - xv) + f \end{aligned} \tag{2}$$

where

$$f(x,y) = 1 + z = 1 - \sqrt{1-x^2-y^2} = \frac{1}{2}(x^2+y^2) + \frac{1}{8}(x^2+y^2)^2 + \dots.$$

The equilibrium point is at the origin and the value of the Hamiltonian there is $H(0,0,0,0) = -1$. To study the dynamics near the equilibrium point we will restrict to the energy level $H = -1 + \frac{1}{2}\epsilon^2$ where $\epsilon > 0$ is a small parameter. Rescale the variables by substituting $(x,u,y,v) \mapsto \epsilon(x,u,y,v)$. Then the blown-up variables satisfy Hamilton's equations for the Hamiltonian function

$$\begin{aligned} \tilde{H}(x,u,y,v,\epsilon) &= \epsilon^{-2}(H(\epsilon x, \epsilon u, \epsilon y, \epsilon v) + 1) \\ &= H_2(x,u,y,v) + \epsilon H_3(x,u,y,v) + \epsilon^2 H_4(x,u,y,v) + \dots \end{aligned}$$

where $H_d(x,u,y,v)$ gives the terms of degree d in the Taylor expansion of H. The quadratic part of the Hamiltonian is

$$H_2 = \frac{1}{2}(u^2+v^2+x^2+y^2) + \rho\sin\lambda(yu - xv) \tag{3}$$

and we have

$$\begin{aligned} H_3 &= \frac{1}{2}\rho\cos\lambda\left((x^2-y^2)v - 2xyu\right) \\ H_4 &= \frac{1}{8}(x^2+y^2)^2 - \frac{1}{2}(xu+yv)^2. \end{aligned} \tag{4}$$

We are interested in the dynamics on the energy manifold

$$\mathcal{M}(\rho,\epsilon) = \{\tilde{H} = \frac{1}{2}\}.$$

For $(\rho,\epsilon) = (0,0)$ this reduces to the unit three-sphere

$$S^3 = \{x^2 + u^2 + y^2 + v^2 = 1\}$$

and it is easy to see that for (ρ,ϵ) sufficiently small there is a diffeomorphism $\mathcal{M}(\rho,\epsilon) \simeq S^3$. Later we will take the point of view that we are studying a family of flows on S^3.

From the quadratic part of the Hamiltonian (6) we find that the linearized differential equations for (x,u,y,v) have matrix

$$A = \begin{bmatrix} 0 & 1 & \rho\sin\lambda & 0 \\ -1 & 0 & 0 & \rho\sin\lambda \\ -\rho\sin\lambda & 0 & 0 & 1 \\ 0 & -\rho\sin\lambda & -1 & 0 \end{bmatrix}.$$

When $\rho = 0$ we have a two-dimensional harmonic oscillator with eigenvalues $\pm i, \pm i$. For $\rho > 0$ the eigenvalues are distinct $\pm i(1+\rho\sin\lambda), \pm i(1-\rho\sin\lambda)$. For small ρ, the quadratic form H_2 is positive definite and so the equilibrium point is linearly and nonlinearly stable, as expected.

The linearized differential equations admit two families of periodic solutions with angular velocities $1 \pm \rho\sin\lambda$. We will begin the local analysis by making a linear change of variables to put these periodic solutions in coordinate planes. Define new variables (x_1, y_1, x_2, y_2) by

$$\begin{bmatrix} x \\ u \\ y \\ v \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}. \tag{5}$$

The quadratic part of the Hamiltonian becomes

$$\begin{aligned} H_2(x_1,y_1,x_2,y_2) &= \frac{1}{2}(1-\rho\sin\lambda)(x_1^2+y_1^2) + \frac{1}{2}(1+\rho\sin\lambda)(x_2^2+y_2^2) \\ &= (1-\rho\sin\lambda)I_1 + (1+\rho\sin\lambda)I_2 \end{aligned} \tag{6}$$

where, for convenience, we have introduced the action variables

$$I_1 = \frac{1}{2}(x_1^2+y_1^2) \qquad I_2 = \frac{1}{2}(x_2^2+y_2^2).$$

Now the two families of periodic solutions lie in coordinate planes. The (x_1,y_1)-plane is filled with periodic solutions of angular velocity $1-\rho\sin\lambda$. It is easy to check that in the original pendulum variables, these orbits move counterclockwise on circles in the (x,y)-plane. Similarly, the (x_2,y_2)-plane is filled with periodic solutions of angular velocity $1+\rho\sin\lambda$ which move in clockwise circles in the (x,y)-plane.

To study this system for ϵ, ρ small, we will put the low-order terms in Birkhoff normal form with respect to the harmonic oscillator Hamiltonian

$$H_{harm} = \frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2).$$

This is a resonant quadratic Hamiltonian whose flow is just a simultaneous rotation of the (x_1, y_1) and (x_2, y_2) planes:

$$(g(t)(x_1, y_1), g(t)(x_2, y_2) \tag{7}$$

where $g(\theta) \in SO(2)$ is the clockwise rotation of the plane by θ radians. Note that the quadratic Hamiltonian (6) is invariant under this flow, even when $\rho \neq 0$.

According to resonant normal form theory [8], we can put all of the low-order terms into rotation-invariant form by means of some further symplectic transformations. As is well-known, it is possible in this way to replace the successive terms $H_3, H_4, \ldots$ by their averages under the rotation. One can check that any odd-degree polynomial has zero average. In particular, it is possible to eliminate H_3 by a symplectic transformation. We omit this calculation but note that since H_3 has a factor of $\rho \cos \lambda$, the required symplectic transformation differs from the identity by terms of order $O(\epsilon\rho)$.

In addition to eliminating the terms of degree 3, this change of coordinates alters the higher-order terms of the Hamiltonian. However, the new fourth-degree terms are necessarily of order $O(\rho^2)$ so H_4 is unchanged to this order. To find the fourth-order terms in the normal form we just need to make our linear change of variables in H_4 from (4) and then average over the rotation (7). After some calculation one finds that the average can be written

$$\begin{aligned}
\overline{H}_4 &= \frac{\epsilon^2}{64}\left(3(x_1^2 + y_1^2 - x_2^2 - y_2^2)^2 - (x_1^2 + y_1^2 + x_2^2 + y_2^2)^2\right) \\
&= \frac{\epsilon^2}{16}\left(3(I_1 - I_2)^2 - (I_1 + I_2)^2\right)
\end{aligned}$$

Since H_{harm} is resonant, one might have expected $\overline{H}_4$ to be more complicated. The fact that it depends only on I_1, I_2 makes the subsequent analysis simpler.

Finally, another symplectic map can be used to eliminate all terms of degree 5. We arrive at a Hamiltonian for the pendulum of the form

$$\tilde{H} = H_{nf} + O(\epsilon^4) + O(\rho^2)$$

where the normal form Hamiltonian is

$$H_{nf} = (1 - \rho \sin \lambda)I_1 + (1 + \rho \sin \lambda)I_2 + \frac{\epsilon^2}{16}\left(3(I_1 - I_2)^2 - (I_1 + I_2)^2\right). \tag{8}$$

We are interested in studying the dynamics of the Hamiltonian system defined by $\tilde{H}$ on the energy level $\tilde{H} = \frac{1}{2}$ for ρ, ϵ small. In the next section, we will have a look at the rather simple behavior of the normal form Hamiltonian H_{nf}.

3. Dynamics of the normal form system

Ignoring the $O(\epsilon^4)$ and $O(\rho^2)$ terms of $\tilde{H}$ gives the polynomial Hamiltonian H_{nf} of (8 and the corresponding Hamiltonian system is integrable. In this section we will exploit this to reduce the dynamics to the study of an integrable twist map of an annulus.

Hamilton's equations for H_{nf} are

$$\begin{aligned} \dot{x}_1 &= \omega_1 y_1 \qquad & \dot{y}_1 &= -\omega_1 x_1 \\ \dot{x}_2 &= \omega_2 y_2 \qquad & \dot{y}_2 &= -\omega_2 x_2. \end{aligned}$$

where

$$\begin{aligned} \omega_1(I_1, I_2) &= 1 - \rho \sin\lambda + \frac{\epsilon^2}{4}(I_1 - 2I_2) \\ \omega_2(I_1, I_2) &= 1 + \rho \sin\lambda + \frac{\epsilon^2}{4}(I_2 - 2I_1). \end{aligned} \tag{9}$$

As before, the action variables are $I_j = \frac{1}{2}(x_j^2 + y_j^2) = \frac{1}{2}r_j^2$,

There are simple periodic solutions in the (x_j, y_j) coordinate planes. For any radius r_1 we have a solution

$$x_1 = r_1 \cos(\nu_1 t) \qquad y_1 = -r_1 \sin(\nu_1 t) \qquad x_2 = y_2 = 0 \tag{10}$$

where the angular velocity is given by

$$\nu_1 = \omega_1(I_1, 0) = 1 - \rho \sin\lambda + \epsilon^2 r_1^2/8.$$

The characteristic multipliers of these orbits are easily found to be

$$\mu = 1, 1, e^{2\pi i \delta_1/\nu_1}, e^{-2\pi i \delta_1/\nu_1} \qquad \delta_1 = \rho \sin\lambda - \frac{3\epsilon^2 r_1^2}{16}.$$

Back in the original pendulum variables, (x, y), these orbits represent approximately circular, counterclockwise motions of the pendulum for ϵ, ρ sufficiently small. The repeated multiplier 1 is due to the Hamiltonian nature of the differential equation. If we fix the value of the energy H_{nf} and consider a Poincaré map defined by a cross-section to the periodic orbit, then the periodic orbit determines a fixed point with multipliers $e^{\pm 2\pi i \delta_1/\nu_1}$ and the orbits are nondegenerate.

Similarly, there are periodic orbits of the form

$$x_2 = r_2 \cos(\nu_2 t) \qquad y_2 = -r_2 \sin(\nu_2 t) \qquad x_1 = y_1 = 0 \tag{11}$$

where

$$\nu_2 = \omega_2(0, I_2) = 1 + \rho \sin\lambda + \epsilon^2 r_1^2/8$$

and the multipliers are

$$\mu = 1, 1, e^{2\pi i \delta_2/\nu_2}, e^{-2\pi i \delta_2/\nu_2} \qquad \delta_2 = \rho \sin\lambda + \frac{3\epsilon^2 r_2^2}{16}.$$

For small ϵ, ρ these orbits represent approximately circular, clockwise motions of the pendulum in the (x, y)-plane.

On the complement of these periodic orbits, it is convenient to use action-angle variables (I_j, θ_j) where $I_j = \frac{1}{2}(x_j^2 + y_j^2) = \frac{1}{2}r_j^2$ and θ_j is the *clockwise* angle in the (x_j, y_j)-plane. Since these are symplectic coordinates, Hamilton's equations give

$$\begin{aligned} \dot{I}_1 &= 0 \quad \dot{\theta}_1 = \omega_1(I_1, I_2) \\ \dot{I}_2 &= 0 \quad \dot{\theta}_2 = \omega_2(I_1, I_2). \end{aligned} \tag{12}$$

Fixing $I_1 > 0, I_2 > 0$ gives an invariant two-dimensional torus with a Kronecker flow with angular velocites (ω_1, ω_2).

Let $\mathcal{N}(\rho, \epsilon) = \{H_{nf} = \frac{1}{2}\}$ denote the energy manifold for H_{nf}. The energy equation depends only on the action variables:

$$(1 - \rho \sin\lambda)I_1 + (1 + \rho \sin\lambda)I_2 + +\frac{\epsilon^2}{16}\left(3(I_1 - I_2)^2 - (I_1 + I_2)^2\right) = \frac{1}{2}.$$

Because of this, we can construct a diffeomorphism $\mathcal{N}(\rho, \epsilon) \simeq S_3$ which preserves the angular variables. More precisely, let $\xi_1, \eta_1, \xi_2, \eta_2$ be new variables and let $J_1 = (\xi_1^2 + \eta_1)^2/2, J_2 = (\xi_2^2 + \eta_2^2)/2$. We will find positive functions $\alpha_1(J_1, J_2), \alpha_2(J_1, J_2)$ such that the substitutions

$$(x_1, y_1) = \alpha_1(J_1, J_2)(\xi_1, \eta_1) \qquad (x_2, y_2) = \alpha_2(J_1, J_2)(\xi_2, \eta_2) \tag{13}$$

reduce the energy equation to $J_1 + J_2 = \frac{1}{2}$, the equation of the unit sphere S^3. The action variables satisfy $I_j = \alpha_j^2 J_j$ and the form of the energy equation suggests setting

$$\alpha_1^2 = (1 - \rho \sin\lambda)^{-1}(1 + \epsilon^2\alpha) \qquad \alpha_2^2 = (1 + \rho \sin\lambda)^{-1}(1 + \epsilon^2\alpha)$$

where $\alpha(J_1, J_2, \rho, \epsilon)$ is a real-valued analytic function to be determined. Substituting this into the energy equation and setting $J_1 + J_2 = \frac{1}{2}$ gives an equation for α of the form

$$\frac{1}{2}\alpha + \frac{1}{16}(3(J_1 - J_2)^2 - \frac{1}{4}) + O(\rho) + O(\epsilon^2) = 0$$

and then the implicit function theorem shows that such a function α exists.

The change of variables (13) gives a diffeomorphism $F : S^3 \to \mathcal{N}(\rho, \epsilon)$. Although this diffeomophism is not symplectic we can still use it to pull back the flow on $\mathcal{N}(\rho, \epsilon)$ to S^3. The differential equations have the same simple structure as before. The three-sphere is given by $J_1 + J_2 = \frac{1}{2}$ or $\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2 = 1$. Fixing $J_1 > 0, J_2 > 0$ gives an invariant torus and the periodic orbits of (10) and (11) have been moved to the unit circles in the (ξ_1, η_1) and (ξ_2, η_2) planes, respectively. Since $J_k = I_k + O(\rho) + O(\epsilon^2)$, the formulas for the angular velocities $\omega_i(J_1, J_2)$ of the corresponding tori and periodic orbits are given, up to higher order terms, by (9) with I_k replaced by J_k.

We are now in a position to reduce the flow of the normal form Hamiltonian to an integrable twist map of the annulus. The idea of reducing the dynamics to an annulus map goes back to Poincaré, where it motivated the formulation of the Poincaré-Birkhoff fixed point theorem [9, 3, 4]. The construction here, based on the existence of two simple periodic orbits, is similar to the one used by Birkhoff [2] and Conley [6] for the restricted three-body problem.

Focus attention on the periodic orbits represented by the unit circles in the (ξ_1, η_1) and (ξ_2, η_2) planes. We can visualize S^3 via a stereographic projection taking the two periodic orbits to linked circles in $\mathbb{R}^3$ as shown in Figure 3. Define a two-dimensional surface $\Sigma \subset S^3$ by

$$\Sigma = \{\xi_1\eta_2 + \xi_2\eta_1 = 0, \xi_1\xi_2 + \eta_1\eta_2 \geq 0\}. \tag{14}$$

Σ consists of the two special periodic orbits, together with the complementary points whose angle variables satisfy $\psi = \theta_1 + \theta_2 \equiv 0 \bmod 2\pi$. It is diffeomorphic to a closed annulus with the periodic orbits playing the role of the boundary circles. Figure 3 shows the stereographic projection of Σ. As coordinates on the annulus we could use $J_1 \in [0, \frac{1}{2}]$ and $\theta_1 \bmod 2\pi$ or $J_2, -\theta_2$.

The invariant tori given by fixing the action variables J_1, J_2 intersect the annulus Σ in invariant circles. The Poincaré map acts as a rigid rotation on each circle and it is easy to find the rotation angle. We have

$$\dot{\psi} = \dot{\theta}_1 + \dot{\theta}_2 + \omega_1(J_1, J_2) + \omega_2(J_1, J_2) = 2 - \frac{\epsilon^2}{8} + O(\rho\epsilon^2) + O(\epsilon^4)$$

where we used (9) and the equation of S^3, $J_1 + J_2 = \frac{1}{2}$. It follows that the return time to Σ satisfies

$$T_{ret} = \pi + \frac{\pi\epsilon^2}{16} + O(\rho\epsilon^2) + O(\epsilon^4).$$

If we use (J_1, θ_1) as coordinates on the annulus we find that the Poincaré map $\Phi_{nf} : \Sigma \to \Sigma$ is given by

$$\begin{aligned} \Phi_{nf}(J_1, \theta_1) &= (J_1, \theta_1 + \omega_1(J_1, \frac{1}{2} - J_1)T_{ret}) \\ &= (J_1, \theta_1 + g(J_1, \rho, \epsilon)). \end{aligned} \tag{15}$$

where

$$g(J_1, \rho, \epsilon) = \pi\left(1 - \rho\sin\lambda + \frac{3\epsilon^2}{16}(4J_1 - 1)\right) + O(\rho\epsilon^2) + O(\epsilon^4). \tag{16}$$

The boundary circles of the annulus, given by $J_1 = 0$ and $J_1 = \frac{1}{2}$, represent the clockwise and counterclockwise nearly circular periodic orbits (11) and (10) respectively. The Poincaré map Φ_{nf} extends continuously to these circles and rotates them by angles $g(0, \rho, \epsilon), g(\frac{1}{2}, \rho, \epsilon)$ where, up to terms of higher order,

$$g(0, \rho, \epsilon) = \pi\left(1 - \rho\sin\lambda - \frac{3\epsilon^2}{16}\right) \qquad g(\frac{1}{2}, \rho, \epsilon) = \pi\left(1 - \rho\sin\lambda + \frac{3\epsilon^2}{16}\right)$$

Moreover, for $\epsilon \neq 0$ and ρ, ϵ sufficiently small, the rotation angle $g(J_1, \rho, \epsilon)$ increases monotonically between these limits as J_1 runs over $[0, \frac{1}{2}]$. Thus, for these parameter values, we have reduced the normal form dynamics to an integrable, monotone twist map of the annulus Σ.

In the next section we will see that the dynamics of the full Foucault Hamiltonian can also be reduce to a monotone twist map, not necessarily integrable. Before discussing the perturbation, however, we will give an intuitive interpretation of the integrable map Φ_{nf}.

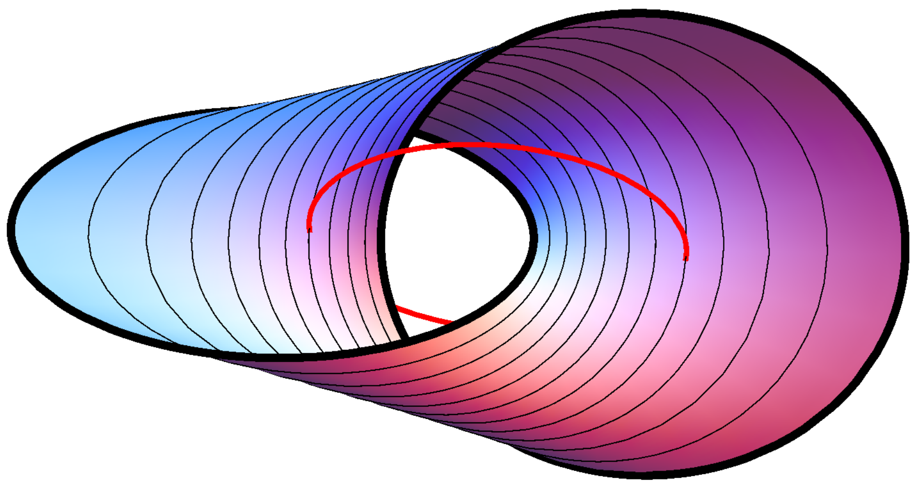

FIGURE 3. Annular surface of section in a stereographic projection of the three-sphere. The annulus is bounded by nearly circular periodic orbits. The interior is a surface of section for the flow and the Poincaré map extends smoothly to the boundary circles. For the normal form system the Poincaré map is an integrable twist map and the annulus is foliated by invariant curves. Also shown is a typical orbit of the harmonic oscillator Hamiltonian.

First consider the case $(\rho, \epsilon) = (0, 0)$. The Hamiltonian reduces to that of the two-dimensional harmonic oscillator which describes the linearized motion of a spherical pendulum with no rotation of the planet. The energy manifold is S^3 and every orbit in it is periodic of period 2π. The orbits are the circles of the Hopf fibration of S^3. Every orbit traces out an ellipse in the (x, y) plane (the ellipse reduces to a line segment for the planar motions of the pendulum). The Poincaré section Σ selects the points where the noncircular ellipses reach their maximum distance from the origin. This happens twice in each period so each Hopf circle intersects the annulus in two points (see Figure 3). The Poincaré map just rotates the annulus rigidly by π and the second iterate is the identity map of the annulus. Thus there is no twist in this case.

Next suppose $\rho > 0$ but $\epsilon = 0$. This describes the linearized dynamics on a rotating planet. The energy manifold is an ellipsoid which we have pulled back to S^3 by a change of variables. There are still two circular periodic orbits but their periods are slightly more or slightly less than 2π depending on whether they rotate in the same or opposite direction to the planet. If we project the noncircular orbits in S^3 to the (x, y) plane, the Poincaré map still gives the points where the orbits are at maximal distance from the origin. This time the Poincaré map is a rigid rotation of the annulus by angle $\pi(1-\rho \sin \lambda)$. Thus the sequence of Poincaré points is slowly precessing with the precessing rate proportional to $\rho \sin \lambda$ as predicted by the simplest explanations of the Foucault pendulum. All the circles are rotated by the same angle, so there is no twist.

The parameters $\rho = 0$, $\epsilon \neq 0$ but small, describe the small oscillations of a nonlinear, spherical pendulum on a nonrotating planet. There are still two circular periodic orbits of period 2π and the planar orbits remain planar. However, the nonlinearity produces a small precession of the nonplanar, noncircular orbits. The precession is in the same direction as the motion of the pendulum. In other words, a pendulum close to a clockwise elliptical orbit will precess slowly in the clockwise sense and similarly, counterclockwise orbits precess counterclockwise. The precession rate is proportional to ϵ^2 and has opposite signs on the two boundary circles of the annulus producing a twist.

The general case of the normal form dynamics, where both ρ and ϵ are nonzero, can be understood as a superposition of the two previous cases. The orbits are approximately elliptical but there are two sources of precession, the rotation of the planet and the nonlinearity of the pendulum itself. Depending on the sign of $\rho \sin \lambda$ and the sense of rotation around the approximate ellipse, these two precessions can be in the same or opposite directions. Suppose for example that $\rho \sin \lambda > 0$ as would be the case for a Foucault pendulum in the northern hemisphere on the earth. Since the earth is rotating counterclockwise under the pendulum, one expects it to produce a clockwise precession and this is reflected by the negative sign on $\rho \sin \lambda$ in (16) for the Poincaré map. The orbits tracing approximate clockwise ellipses experience an additional clockwise precession due to the nonlinearity but for those near counterclockwise elliptical motions the nonlinear precession is counterclockwise. For the latter, the net direction of precession of these orbits will depend on the relative sizes of the two effects.

Suppose for example that

$$-\frac{3\epsilon^2}{16} < \rho \sin \lambda < \frac{3\epsilon^2}{16}.$$

Then near the $J_1 = 0$ boundary of the Σ, the rotation angle is a bit less than π while near the $J_1 = \frac{1}{2}$ boundary, it is a bit more than π. This means that for nearly circular counterclockwise motions the net precession is counterclockwise – the nonlinear effect more than compensates for the rotation of the earth. Moreover, for these parameter values there will be orbits which exhibit no precession at all. Indeed there will be a unique value $J_1^* \in [0, \frac{1}{2}]$ such that $g(J_1^*, \rho, \epsilon) = \pi$. In fact,

$$J_1^* \approx \frac{1}{4} + \frac{4\rho \sin\lambda}{3\epsilon^2} = \frac{1}{4} + \frac{4\kappa}{3} \qquad \text{where } \kappa = \frac{\rho\sin\lambda}{\epsilon^2}.$$

This means that there is a circle of periodic points of period 2 for the Poincaré map Φ_{nf} or equivalently a torus filled with periodic orbits of period approximately 2π for the Hamiltonian flow. We will see that at least two of these orbits persist when we perturb to the full Foucault Hamiltonian. Note however, that for orbits with $J_1 = \frac{1}{4}$, the precession rate reduces to $\pi(1 - \rho\sin\lambda)$, as for the linearized pendulum. Clearly the precessional behavior depends, even qualitatively, on the initial conditions of the pendulum.

Foucault's pendulum in the Pantheon in Paris had a length approximately $67\,m$ and was moving with an amplitude of about $3\,m$ so the variables (x, y) on the unit sphere were oscillating with amplitude about $3/67$ and the energy was $H = -1 + \frac{1}{2}\epsilon^2$ where $\epsilon \approx 3/67 \approx 0.0448$. We already saw that $\rho \approx 0.00019$ and the latitude of Paris is about $\lambda = 48.85°$. Hence

$$0 < \frac{\rho\sin\lambda}{\epsilon^2} = \kappa \approx 0.0713 < \frac{3}{16}$$

and we see that Foucault's parameters fall into the range described in the last paragraph.

To see what a nonprecessing orbit might look like using the Foucault parameters we will relate the values of the action variables (J_1, J_2) to the shape of the orbit in the (x, y) plane. Since all of the orbits considered here are small perturbations of orbits of the harmonic oscillator, we will assign to each pair of action variables a reference ellipse giving the shape of the corresponding orbit of the oscillator. From the linear change of variables (5) one finds that for the ellipse associated to (J_1, J_2), the semimajor axes have ratio

$$\frac{b}{a} = \frac{\delta}{1 + \sqrt{1 - \delta^2}} \qquad \delta = \frac{J_1 - J_2}{J_1 + J_2}$$

where the ratio is positive or negative depending on whether the elliptical orbit is clockwise or counterclockwise, respectively. For the nonprecessing orbits we have $\delta^* = \frac{16\kappa}{3}$. For Foucault's parameters we have $\delta^* \approx 0.38$ which gives an axis ratio of $(b/a)^* \approx 0.2$. Figure 4 shows an ellipse with this axis ratio. Based on the normal form approximation, a counterclockwise motion of Foucault's pendulum with this shape would not precess. Of course motions used in Foucault's demonstration are much closer to planarity.

Smaller Foucault pendula can lead to more disturbing results. For example, a pendulum of length $24\,m$ and moving with an amplitude of $2\,m$ at latitude $\lambda = 41.2^\circ$ has parameter $\kappa \approx 0.01$ and so the shape of a nonprecessing motion has axis ratio of only $(b/a)^* \approx 0.03$. As Figure 4 shows, this is almost planar.

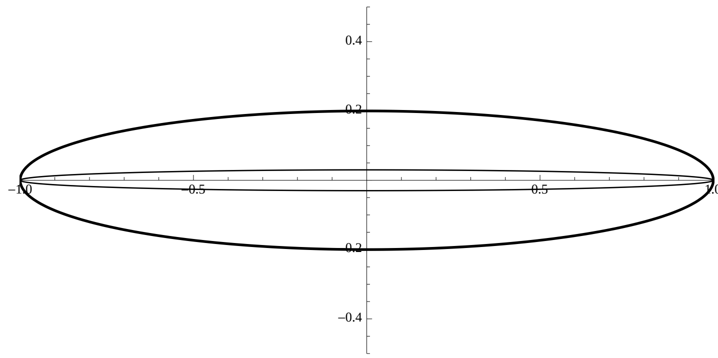

FIGURE 4. Approximate shapes of nonprecessing periodic orbits. If Foucault's original Pantheon pendulum were set in motion counterclockwise around the outer ellipse, it would not precess due to nonlinear effects. For smaller pendula, even a nearly planar motion along the inner ellipse can halt the precession.

4. A twist map for the Foucault pendulum problem

Recall that our Hamiltonian for the Foucault pendulum is of the form

$$\tilde{H} = H_{nf} + O(\rho^2) + O(\epsilon^4).$$

The goal of this section is to show that some of the features of the normal form dynamics persist under perturbation. In particular, we will see that the dynamics can be reduced to a monotone twist map $\Phi : \Sigma \to \Sigma$ which is a perturbation of the integrable map Φ_{nf} studied in the last section.

First we will see that the two nearly circular orbits corresponding to the boundaries of the annulus Σ can be continued to the full problem. The presence of two perturbation parameters presents a technical difficulty. To handle this we will use the parameter κ introduced above, such that

$$\rho \sin \lambda = \kappa \epsilon^2.$$

If we fix κ we are restricting the parameters to a parabolic curve through the origin in parameter space. The Hamiltonian of the pendulum takes the form

$$\tilde{H}_{\kappa,\epsilon} = I_1 + I_2 + \frac{\epsilon^2}{16}\left(16\kappa(I_2 - I_1) + (I_1 + I_2)^2 - 3(I_1 - I_2)^2\right) + O(\epsilon^4).$$

Theorem 1. *Fix $\kappa \neq \frac{3}{16}$. Then there is $\epsilon_0(\kappa) > 0$ and an analytic family of periodic solutions for the Hamiltonian $\tilde{H}_{\kappa,\epsilon}$, $0 \leq \epsilon < \epsilon_0(\kappa)$ with energy*

$\frac{1}{2}$ which converges to the counterclockwise circular solution of radius 1 as $\epsilon \to 0$. If $\kappa \neq -\frac{3}{16}$ there is a similar family of clockwise periodic solutions.

Proof. Let $\mathcal{M}(\kappa,\epsilon) = \{\tilde{H}_{\kappa,\epsilon} = \frac{1}{2}\}$ denote the energy manifold of interest and let $\mathcal{N}(\kappa,\epsilon) = \{H_{nf} = \frac{1}{2}\}$ be the corresponding energy manifold for the normal form Hamiltonian. In the last section we constructed a diffeomorphism $F : S^3 \to \mathcal{N}(\rho,\epsilon)$ of a special form (13). Since the true Hamiltonian differs from H_{nf} by terms of order $O(\epsilon^4)$ there will be another diffeomorphism $G : \mathcal{N}(\kappa,\epsilon) \to \mathcal{M}(\kappa,\epsilon)$ which is the identity to this order. The composition gives a diffeomorphism $G \circ F : S^3 \to \mathcal{M}(\kappa,\epsilon)$ and $G \circ F = F + O(\epsilon^4)$.

If we use this diffeomorphism to pull back the differential equations on $\mathcal{M}(\kappa,\epsilon)$ to S^3, the resulting differential equations for the variables ξ_j, η_j or the corresponding action-angle variables J_k, θ_k differ from the pull-backs of the normal form differential equations by terms of order $O(\epsilon^4)$. Of course, they also agree with the equations for the harmonic oscillator up to terms of order $O(\epsilon^2)$.

Now that all of the flows have been pulled back to S^3, set up a Poincaré section near the counterclockwise circular orbit for $\epsilon = 0$ represented by the unit circle in the (ξ_1,η_1)-plane. For example we could choose the intersection of S^3 with the half-plane determined by $\eta_1 = 0, \xi_1 > 0$ and we could use $z = (\xi_2,\eta_2)$ as local coordinates in this section. Let $\phi_\epsilon(z)$ denote the Poincaré map for the flow corresponding the $\tilde{H}_{\kappa,\epsilon}$ and $\psi_\epsilon(z)$ that determined by the flow of H_{nf}. Both ϕ_ϵ and ψ_ϵ are equal to the identity map to order $O(\epsilon^2)$. Furthermore, $\phi_\epsilon = \psi_\epsilon + O(\epsilon^4)$.

By construction, the origin is a fixed point of ψ_ϵ and the formulas in the last section show that its multipliers are $e^{\pm 2\pi i \delta_1/\nu_1}$ where

$$\delta_1 = \rho \sin\lambda - \frac{3\epsilon^2}{16} + O(\epsilon^4) = \epsilon^2(\kappa - \frac{3}{16}) + O(\epsilon^4).$$

These multipliers are of the form $1 \pm \epsilon^2 2\pi i(\kappa - \frac{3}{16}) + O(\epsilon^4)$ and it follows that the Poincaré map ϕ_ϵ can be written

$$\phi_\epsilon(z) = z + \epsilon^2 Lz + O(\epsilon^2|z|^2) + O(\epsilon^4)$$

where L is a 2×2 matrix with eigenvalues $\pm 2\pi i(\kappa - \frac{3}{16})$.

Now the fixed point equation is $\phi_\epsilon(z) - z = 0$. We can divide this equation by ϵ^2 to obtain an equation of the form

$$g(z,\epsilon) = Lz + O(|z|^2) + O(\epsilon^2) = 0.$$

Now apply the implicit function theorem near $(z,\epsilon) = (0,0)$ to get an analytic family of fixed points $z(\epsilon)$. By construction these lie in the required energy levels and are near the counterclockwise circular solutions.

A similar proof works for the clockwise family. □

From now on we assume $\kappa \neq \pm\frac{3}{16}$. Then on each energy manifold $\mathcal{M}(k,\epsilon)$, for ϵ sufficiently small, there will be two such periodic orbits, one counterclockwise and one clockwise. After pulling the flow back to S^3 we have two families of nearly circular periodic orbits, say $\gamma_1(\epsilon)$ and $\gamma_2(\epsilon)$ which converge to the unit circles in the (ξ_1, η_1) and (ξ_2, η_2) planes as $\epsilon \to 0$. In the next paragraph we will describe how to construct an analytic diffeomorphism of S^3 which moves these periodic orbits exactly onto these unit circles. After applying this additional diffeomorphism, we obtain a flow on S^3 which leaves the two unit circles invariant. Then we can use same annular surface of section Σ, given by (14), as we did for the normal form.

To describe the construction of the required diffeomorphism, let $z_j = (\xi_j, \eta_j) \in \mathbb{R}^2$ and $z = (z_1, z_2) \in \mathbb{R}^4$ and let $|z_j|, |z|$ denote the Euclidean norms. The counterclockwise orbit can be written as a graph over the unit circle $z_2 = \epsilon^4 \alpha(z_1, \epsilon), |z_1| = 1$ where $\alpha : S^1 \to \mathbb{R}^2$ is real-analytic. We can extend α to a real-analytic function $\tilde{\alpha} : D \to \mathbb{R}^2$ on the unit disk with the property that $\tilde{\alpha}(0,0) = (0,0)$. For example, we can find a real-analytic harmonic extension by solving Laplace's equations with boundary condition α and then add a suitable multiple of $1 - |z_1|^2$ to get the right value at the origin. Then the diffeomorphism $(z_1, z_2) \mapsto (z_1, z_2 - \epsilon^4 \tilde{\alpha}(z_1))$ maps the counterclockwise orbit into the z_1 plane while fixing the z_2 plane. Applying a similar diffeomorphism then moves the clockwise orbit into the z_2 plane fixing the z_1 plane. We can compose with a radial projection $z \mapsto z/|z|$ along each ray to get a diffeomorphism $S^3 \to S^3$ moving the two orbits into the coordinate planes and it follows that the orbits then map to the unit circles in these planes.

Now that the periodic orbits are the unit circles, we can use Σ as is (14) for a cross-section. The invariance of the circles implies that the differential equation for the angular variable θ_j extends smoothly to $J_k = 0$ and they will differ from the corresponding equations for the normal form by terms of order $O(\epsilon^4)$. In particular, we have

$$\dot{\psi} = \dot{\theta}_1 + \dot{\theta}_2 = 2 - \frac{\epsilon^2}{8} + O(\epsilon^4)$$

for the angle ψ whose vanishing mod 2π characterizes the interior of Σ. Thus the interior of Σ is still a Poincaré section whose return time satisfies

$$T_{ret} = \pi + \frac{\pi\epsilon^2}{16} + O(\epsilon^4).$$

The Poincaré map $\Phi : \Sigma \to \Sigma$ for the Foucault Hamiltonian is of the form

$$\begin{aligned} \Phi(J_1, \theta_1) &= (J_1 + O(\epsilon^4), \theta_1 + g(J_1, \kappa, \epsilon) + O(\epsilon^4)) \\ g(J_1, \kappa, \epsilon) &= \pi \left(1 + \epsilon^2 (\frac{3}{16}(4J_1 - 1) - \kappa) \right). \end{aligned} \tag{17}$$

Here the terms $O(\epsilon^4)$ may depend on both J_1, θ_1 so the map is probably nonintegrable.

Φ is a small boundary-preserving perturbation of the Poincaré map Φ_{nf} studied in the last section. Up to terms of order $O(\epsilon^4)$, it rotates the boundary circles by angles $g(0,\kappa,\epsilon)$, $g(\frac{1}{2},\kappa,\epsilon)$ where

$$g(0,\kappa,\epsilon) = \pi\left(1-\epsilon^2(\kappa+\frac{3}{16})\right) \qquad g(\frac{1}{2},\kappa,\epsilon) = \pi\left(1-\epsilon^2(\kappa-\frac{3}{16})\right). \tag{18}$$

Moreover, for $\epsilon \neq 0$ sufficiently small, $g(J_1,\kappa,\epsilon)$ increases monotonically and so we have once again reduced the dynamics to a monotone twist map of the annulus Σ.

Our diffeomorphism $S^3 \to \mathcal{M}(\kappa,\epsilon)$ maps Σ to a surface Σ' in the energy manifold. By a well-known result in Hamiltonian mechanics, the corresponding Poincaré map preserves the area element induced by the symplectic structure on $\mathbb{R}^4$. So the map Φ preserves the pull-back of this area element under the diffeomorphism. In this sense, Φ is an area-preserving monotone twist map and all of the powerful theorems about such maps apply.

Theorem 2. *Fix $\kappa \neq \pm\frac{3}{16}$. Then there is $0 < \epsilon_1(\kappa) < \epsilon_0(\kappa)$ such that the flow of the Foucault pendulum Hamiltonian $\tilde{H}_{\kappa,\epsilon}$ with energy $\frac{1}{2}$ can be reduced to an area-preserving monotone twist map of an annulus. More precisely, there is an annular surface $\Sigma' \subset \mathcal{M}(\kappa,\epsilon)$ bounded by the periodic orbits of Theorem 1 whose interior is a global surface of section on the complement of these orbits. There are coordinates (J_1,θ_1) on Σ' such that the Poincaré map is an area-preserving monotone twist map of the form (17).*

We will conclude by mentioning some corollaries which follow from the general theory of area-preserving monotone twist maps. Let α,β denote the rotation numbers of Φ on the two boundary circles of the annulus where we view rotation numbers as angles modulo 2π. Up to order $O(\epsilon^4)$ these are given by (18). The Poincaré-Birkhoff theorem implies that for every rational number, $\frac{p}{q}$ with $\alpha < \frac{2\pi p}{q} < \beta$ there are at least two periodic points of Φ of period q with that rotation number. This represents a periodic solution of the Foucault pendulum with period approximately $q\pi$. This is particularly interesting in the case $-\frac{3}{16} < \kappa < \frac{3}{16}$.

Theorem 3. *If $-\frac{3\epsilon^2}{16} < \rho\sin\lambda < \frac{3\epsilon^2}{16}$ and $\epsilon > 0$ is sufficiently small then the Foucault pendulum admits at least two periodic orbits of period approximately 2π. In other words after the pendulum swings back and forth once, it returns to its original state in spite of the rotation of the planet.*

Proof. We have $-\frac{3}{16} < \kappa < \frac{3}{16}$ and from (18) we have $g(0,\kappa,\epsilon) < \pi < g(\frac{1}{2},\kappa,\epsilon)$. It follows that the rotation numbers of the boundary curves satisfy $\alpha < \pi < \beta$ and taking $\frac{p}{q} = \frac{1}{2}$ we see that Φ admits at least two periodic points of period 2. $\square$

The approximate shape of some periodic orbits of this type are shown in Figure 4. As noted already, the parameter κ for the original Foucault pendulum satisfies the hypothesis, although we cannot be sure that the corresponding ϵ is sufficiently small.

In addition to periodic orbits we also get quasi-periodic orbits. Indeed, according to Aubry-Mather theory every rotation number between α and β is realized by some orbit [7]. For rotation numbers whose ratio with 2π is Diophantine, the KAM theory implies that for ϵ sufficiently small Φ admits an invariant curve realizing it [11]. Hence there will be invariant tori for the corresponding flow. In the non-Diophantine case we may get invariant Cantor sets instead.

References

1. G.D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, vol. **9** (1927).
2. G.D. Birkhoff, *The restricted problem of three-bodies*, Rend.Circ.Math. Palermo, **39** (1915) 265–334.
3. G.D. Birkhoff, *Proof of Poincaré's geometric theorem*, Trans.AMS, **14** (1913) 14–22.
4. G.D. Birkhoff, *An extension of Poincaré's last geometric theorem*, Acta Math., **47** (1925) 297–311.
5. J. Chazy, *Sur l'effet pendule sphérique dan l'expérience du pendule de Foucault*, Bull. Soc. Math. Fr., **58** (1930) 100–104.
6. C.C. Conley, *On some new long periodic solutions of the plane restricted three-body problem*, Comm.Pure Appl. Math., **XVI** (1963) 449–467.
7. J. Mather, *Existence of quasi periodic orbits for twist homeomorphisms*, Topology, **21,4** (1963) 457–467.
8. , K.R.Meyer, G.R.Hall, D.Offin, *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, Springer Verlang, New York (2009).
9. H. Poincaré, *Sur le mouvement du pendule conique*, Bull.Sci.Math., **ser.2, 5** (1881) 448–454.
10. M.F. Tisserand, *Sur une théorème de géométrie*, Rend.Circ.Math. Palermo, **33** (1912) 375–407.
11. C.L. Siegel and J. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag,New York (1971).

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