Math 5335 **Classification of Isometries** F'18

Much of this (well, all) is in book, but presented in a different order or with a different viewpoint (e.g. the formula for reflections) So follow these notes instead.

We know from Chapter 4 $\mathcal{U}: \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry iff $\mathcal{U}(X) = MX + P$, $M \in \{R_\theta, F_\theta\}$.

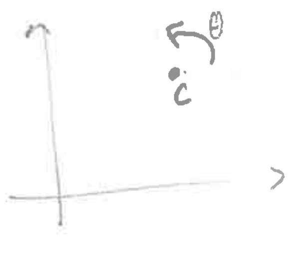
Overriding Questions; How many isometries are there? what are they? How do we know that list is complete?

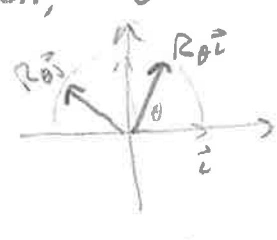
Let's start with a few examples: Do TPD sheets?

Identity $\mathcal{I}(X) = X \quad (= R_0 X + O)$

Transl'n $\mathcal{T}_V(X) = X + V \quad (= IX + V = R_0 X + V)$

Rot'n by θ about C Seems hard in gen'l - so reduce to previously solved problem! By construction, R_θ rotates $\mathbb{R}^2$ by θ about the origin.

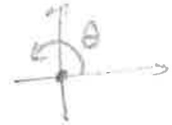




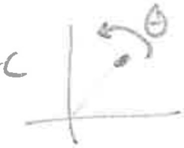
Let's do this in steps!

① Move C to origin: $X-C$

② Rotate by θ about O: $R_\theta(X-C)$



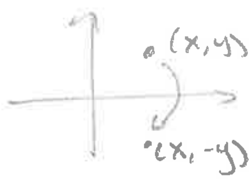
③ Move O back to C $R_\theta(X-C)+C$



$$\boxed{R_{\theta,C}(X) = R_\theta(X-C)+C}$$

Reflections also hard! In steps: across x-axis; across ℓ through O, then across any ℓ.

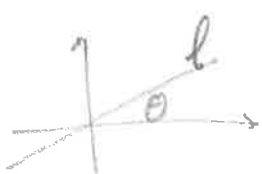
x-axis



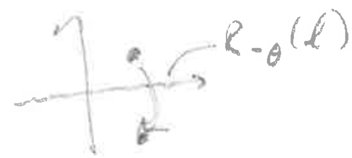
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

ℓ through O



① Rotate by $-\theta$: $R_{-\theta}X$



② Reflect across x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta}X$

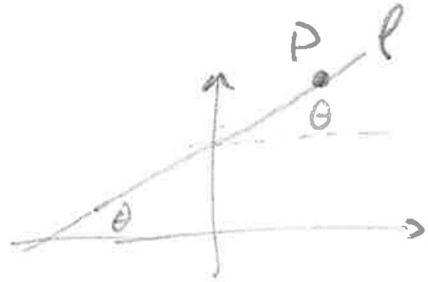
③ Rotate back $R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta}X$

<u>Note:</u> $\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}\right) = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}\begin{bmatrix} c\theta & s\theta \\ s\theta & -c\theta \end{bmatrix}$

$= \begin{bmatrix} c^2\theta - s^2\theta & 2s\theta c\theta \\ 2c\theta s\theta & s^2\theta - c^2\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \underline{F_\theta}$

Thus $T(x) = F_\theta x$ reflects $\mathbb{R}^2$ across line which forms angle of θ w/ x-axis.



<u>General Refl'n across ℓ</u>

(which contains a pt P, forms angle of θ w/ horizontal, measured anywhere.)

(1) Move P to origin: $x - P$

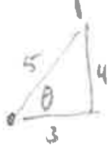
(2) Reflect: $F_\theta(x-P)$

(3) Move O back to P $F_\theta(x-P)+P$

$$\boxed{M_\ell(x) = F_\theta(x-P) + P}$$

<u>Ex</u> Get point $\in$ Quad I, $u = (3,4)$ $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$

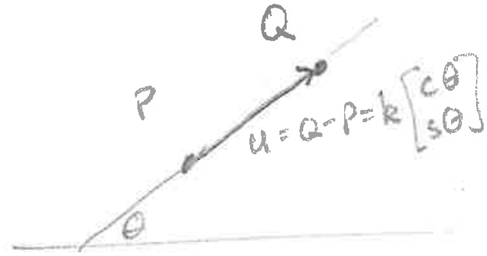
$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25}$



$M_\ell(x) = \begin{bmatrix} -7/25 & 24/25 \\ 24/25 & 7/25 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \ \\ \ \end{bmatrix}\right) + \begin{bmatrix} \ \\ \ \end{bmatrix}$

⚠ Our formula for M_ℓ seems to depend on arbitrary choice of point $P \in \ell$. Uh-oh!

<u>Prop</u> Let $P, Q \in \ell$, which forms angle of θ w/ horizontal. Then



$$F_\theta(X-P)+P = F_\theta(X-Q)+Q$$

(And thus we can use any point of ℓ in formula for M_ℓ).

<u>Pf</u>: Let $U = Q - P = k\begin{bmatrix} c\theta \\ s\theta \end{bmatrix}$. By "Useful Facts" sheet,

$F_\theta U = U$. ⟶ ↓

<u>Method 1</u>

$$F_\theta(Q-P) = Q-P$$

(⚠ both sides!)

$$F_\theta(X - X + Q - P) = Q - P$$

$$F_\theta(X-P-(X-Q)) = Q-P$$

$$F_\theta(X-P) - F_\theta(X-Q) = Q-P$$

$$F_\theta(X-P)+P = F_\theta(X-Q)+Q \quad \blacksquare$$

<u>Method 2</u>

$$\begin{aligned} F_\theta(X-P)+P &= F_\theta(X-Q+Q-P)+P \\ &= F_\theta(X-Q) + F_\theta(Q-P) + P \\ &= F_\theta(X-Q) + Q - P + P \\ &= F_\theta(X-Q) + Q \quad \blacksquare \end{aligned}$$

Combining / Composing Isometries

$\boxed{\text{Do Kaleidescope / Hubcap Activity}}$ day 1 → groups → Leonardo da Vinci

(and any other relevant TPD)

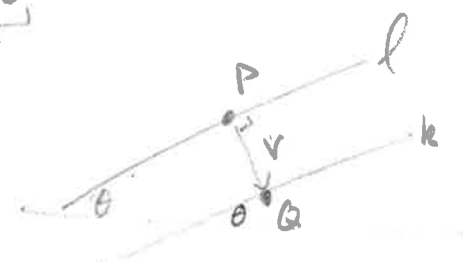
Prop $R_{\varphi,C}\circ R_{\theta,C}(X) = R_{\varphi+\theta,C}(X)$.

Pf

$$R_\varphi\left(\left[R_\theta(X-C)+\cancel{C}\right]-\cancel{C}\right)+C = R_\varphi R_\theta(X-C)+C$$
$$= \underbrace{R_{\varphi+\theta}}_{\text{ch 5}}(X-C)+C$$

(*) Next page

Prop Let $\ell \parallel k$ as shown. Then $M_k\circ M_\ell(X) = T_{2V}(X)$.

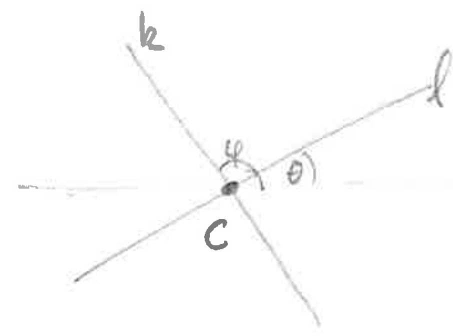


$V = Q-P = k\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$

Pf $M_k(M_\ell(X)) = M_k(F_\theta(X-P)+P)$

$$= F_\theta\left(\left[F_\theta(X-P)+P\right]-Q\right)+Q$$
$$= F_\theta F_\theta(X-P) + F_\theta(P-Q)+Q$$
$$= X+(Q-P)+Q-P$$
$$= X+2(Q-P) = X+2V$$

Prop Let $\ell\cap k=\{C\}$, as shown. Then $M_k\circ M_\ell(X) = R_{2(\varphi-\theta),C}(X)$.



Pf $F_\varphi\left(\left[F_\theta(X-C)+\cancel{C}\right]-\cancel{C}\right)+C = F_\varphi F_\theta(X-C)+C$

$$= \underbrace{R_{2(\varphi-\theta)}}_{GW}(X-C)+C$$

(*)

Also, (maybe do 1st?)

Prop $M_\ell \circ M_\ell = id$, i.e. M_ℓ is an involution.

Pf $F_\theta\left(\left[F_\theta(X-P)+P\right]-P\right)+P = F_\theta F_\theta(X-P)+P$ $\qquad F_\theta F_\theta = I$ (useful facts)

$$= X - P + P$$

$$= X$$

Back to rotations what about $R_{\varphi,D} \circ R_{\theta,C}$ when $C \neq D$?

(Show GeoGebra for $\theta = \varphi = \pi$) looks like a transl'n!

Prop $R_{\pi,D} \circ R_{\pi,C} = T_{2(D-C)}$ $\qquad R_\pi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $R_\pi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$

Pf $R_\pi\left(\left[R_{\pi,C}(X-C)+C\right]-D\right)+D = -(-X+C+C-D)+D$

$$= X + 2D - 2C$$

$$= X + 2(D-C)$$

General Case - harder! (Cut the knot demo?)

↳ No, prove, then GeoGebra Demo.

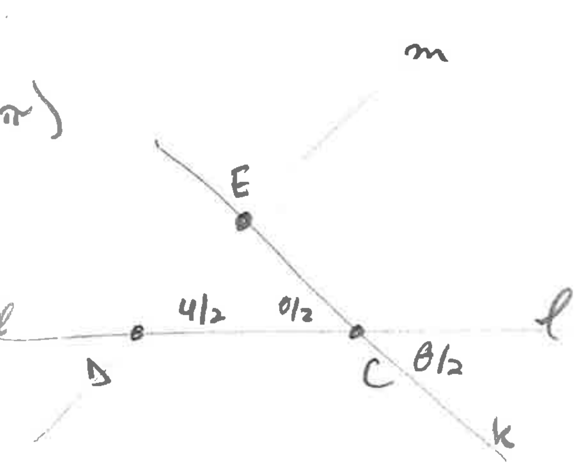
Prop $R_{\varphi,D} \circ R_{\theta,C} = R$

(here $\varphi+\theta \in (-\pi,\pi)$, so $\theta+\varphi \neq 2\pi$)

Pf $\ell = \overleftrightarrow{CD}$, choose k s.t. $R_{\theta,C} = M_\ell \circ M_k$

choose m s.t. $R_{\varphi,D} = M_m \circ M_\ell$

$R \circ R = M_m \circ id \circ M_k$ etc.



Rather than test/try every possible comb'n, need a systematic approach. key:

$\underline{\text{Thm 6.16}}$ Every isometry can be expressed (constructed) as the composition of ≤ 3 refl'ns.

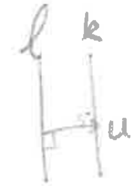
"Pf": Lab 3

$\boxed{\text{Remark}}$ Thus we just need to figure out all possibilities for $n=1,2,3$ refl'ns!

$\underline{n=1}$ M_ℓ refl'n.

$\underline{n=2}$ $M_k \circ M_\ell$ is either:

- $\mathcal{T}_{2u}(X)$ if $\ell \parallel k$:



- $R_{2\theta,c}$ if $\ell \cap k = \{c\}$:

special cases: $\ell = \ell$: $M_\ell \circ M_\ell = \mathrm{id}(X) = \mathcal{T}_0(X) = R_{0,c}(X)$

$\ell \perp k$: $M_k \circ M_\ell = \mathcal{C}_c(X)$ "central inversion"

$(= R_{\pi,c}(X))$

$\underline{n=3}$ $M_k \circ M_\ell \circ M_m = \dots ?$

$\exists$ new possibility ... a glide refl'n!

Def Given $u \parallel \ell$, $\mathcal{G}(X) = \mathcal{G}(X) = M_\ell \circ T_u(X)$ is a <u>glide reflection</u>.

Ex

Prop Given $u \parallel \ell$, $M_\ell \circ T_u = T_u \circ M_\ell$
(so can do transl'n / refl'n in either order.

Pf $M_\ell \circ T_u(X) = F_\theta([X+u]-P)+P$

$= F_\theta X + F_\theta u - F_\theta P + P$

$= F_\theta(X-P)+P + u$

$= T_u(M_\ell(X))$

$F_\theta u = u$! ("Useful Facts")

(Do Glide Refl'ns.)
(Escher Pictures)

How do we know $\mathcal{G}(X)$ isn't a refl'n in hiding? key turns out to be <u>fixed pts</u>.

Def A *fixed pt* of $f: A \to A$ is a pt $a \in A$ s.t. $f(a) = a$.

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 6 - 2x$. If $f(x) = x$, then $6 - 2x = x \Rightarrow x = 2$.

Prop If $u \neq 0$, $\mathcal{T}_u(x)$ has no fixed pts

pf if $x + u = x$, $u = x - x = 0$

Prop Only fixed pt of $\mathcal{R}_{\theta,0}(x)$ is $x = 0$.

pf $\mathcal{R}_{\theta,0}(x) = R_\theta x = x \Rightarrow R_\theta x - x = 0$

$$\underbrace{(R_\theta - I)}_{A}x = 0$$

A is invt'ble (check!)

$x = A^{-1}0 = 0$.

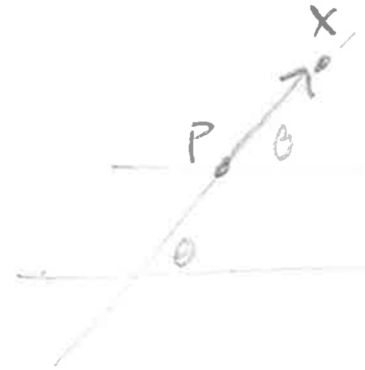
Prop Only fixed pt of $\mathcal{R}_{\theta,c}(x)$ is c.

pf $\mathcal{R}_{\theta,c}(x) = x \Rightarrow R_\theta(x - c) + c = x$.

$R_\theta(x - c) = (x - c) \Rightarrow x - c = 0$ prev. prop

$\Rightarrow x = c$.

Prop Fixed pts of $\mathcal{M}_\ell(X)$ are pts on ℓ



Pf If $X = F_\theta(X-P)+P$, then
$F_\theta(X-P) = (X-P)$. We've
seen (you check) fixed pts/vectors
of F_θ are those $\parallel \ell$. So $X-P$ is DI of ℓ,
and $P+(X-P) = X \in \ell$.

Prop $\mathcal{G}(X)$ is not a transl'n, rot'n, or refl'n. It's
a "new" kind of isometry.

Pf $\mathcal{G}(X) = \mathcal{T}_u(\mathcal{M}_\ell(X)) = F_\theta(X-P)+P+u$.

Since $\mathcal{G}(X) = F_\theta X + (\text{stuff})$ [form $MX+P$], its matrix
is $F_\theta \Rightarrow$ not a transl'n or rot'n.

Suppose $F_\theta(X-P)+P+u = X$ (*)

Rearrange (*): $F_\theta(X-P) = (X-P)+u$

Mult. both sides by F_θ, rearrange: $F_\theta(X-P) = (X-P)-u$.

Thus if $\mathcal{G}(X)$ has fixed pt, $u=-u \Rightarrow u=0 \Rightarrow \mathcal{G}$ a refl'n.

(i.e. if $u=0$, $\mathcal{G}(X)$ not a glide refl'n.

Ok, so $\mathcal{g}(X)$ is a new kind of isometry. Are there any others I can get with 3 refl'ns?

Consider $\underbrace{\mathcal{M}_k \circ \mathcal{M}_\ell \circ \mathcal{M}_m}$

either of these compositions of 2 refl'ns can be rewritten as a transl'n or rotation — or possibly even $\mathcal{U}(X) = \mathcal{T}_0(x) \circ \mathcal{R}_0$.

So we have to consider 4 possibilities:

(1) transl'n $\circ$ reflection
(2) refl'n $\circ$ translation

already know these are GR's *if* transl'n $\parallel$ mirror. We'll have to consider other cases.

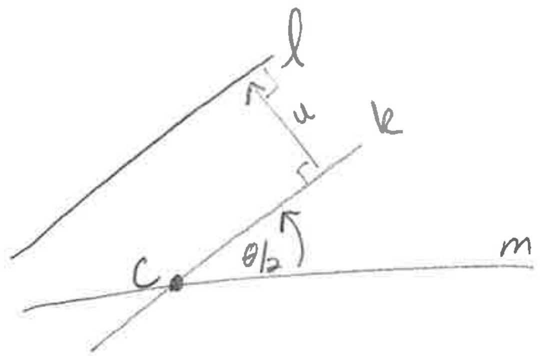
(3) rot'n $\circ$ refl'n
(4) refl'n $\circ$ rot'n

turns out we can show these can be rewritten as composition of transl'n and refl'n, so we don't have to worry about (3) and (4) — they're really the same as (1) and (2), just in hiding!

To see this....

Consider (4): $\mathcal{M}_\ell \circ \mathcal{R}_{\theta,C}$.

$\mathcal{R}_{\theta,C}$ can be constructed by reflecting across two lines which intersect at C, forming an angle of $\theta/2$. Let's choose two lines intersecting at C such that the second line is $\parallel \ell$:



$$\mathcal{M}_\ell \circ \mathcal{R}_{\theta,C} = \mathcal{M}_\ell \circ (\mathcal{M}_k \circ \mathcal{M}_m)$$
$$= \underbrace{(\mathcal{M}_\ell \circ \mathcal{M}_k)} \circ \mathcal{M}_m$$
$$= \mathcal{T}_{2u} \circ \mathcal{M}_m$$

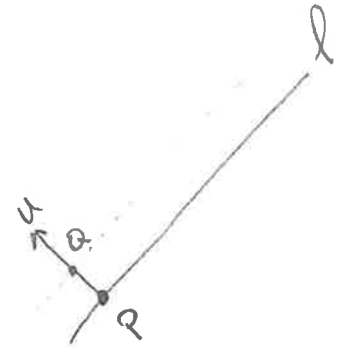
where u is vector from k to ℓ, as shown. Hence (4) can be rewritten in the form of (1).

Similarly, (3) can be rewritten in the form of (2).

Thus we only need to consider composition of refl'n and transl'n, in either order: $\mathcal{T}_u \circ \mathcal{M}_\ell$ or $\mathcal{M}_\ell \circ \mathcal{T}_u$.
If $u \parallel \ell$ this is a glide reflection. What if $u \nparallel \ell$?

1^{st} Case $u \perp \ell$.

Prop Suppose $u \perp \ell$. Then $T_u \circ M_\ell = M_k$, where $k = T_{u/2}(\ell)$.
(i.e. k is the line ℓ, translated by $u/2$).



Pf Let $P \in \ell$, define $Q = P + \frac{1}{2}u$
(so $Q - P = \frac{1}{2}u$ or $u = 2(Q-P)$.)

$$
\begin{aligned}
T_u \circ M_\ell(X) &= T_u\left(F_\theta(X-P)+P\right) \\
&= F_\theta(X-P) + P + u \\
&= F_\theta(X-P) + P + \tfrac{1}{2}u + \tfrac{1}{2}u \\
&= F_\theta(X-P) + P + \tfrac{1}{2}u - \tfrac{1}{2}F_\theta u \\
&= F_\theta(X - P - \tfrac{1}{2}u) + P + \tfrac{1}{2}u \\
&= F_\theta(X - P - Q + P) + P + Q - P \\
&= F_\theta(X - Q) + Q \\
&= M_k(X), \quad \text{for } k = T_{u/2}(\ell).
\end{aligned}
$$

⚠ We know $F_\theta u = -u$, or $u = -F_\theta u$.

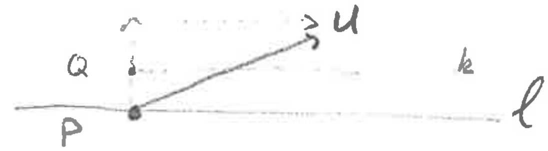
<u>You check:</u> $M_\ell \circ T_u$ (other order) is also a refl'n

2^{nd} Case $\quad u$ not $\parallel$ or $\perp$ ℓ (General Case)

Write u as $u = A + B$, where $A \perp \ell$, $B \parallel \ell$:

Let $P \in \ell$, so $M_\ell(X) = F_\theta(X-P)+P$

We know $F_\theta A = -A$, $\quad F_\theta B = B$.



Let's check $\mathcal{T}_u \circ M_\ell(X)$, which is

$$F_\theta(X-P)+P+u = F_\theta(X-P)+P+\underbrace{\tfrac{1}{2}A+\tfrac{1}{2}A+B}_{u}$$

$$= F_\theta(X-P)+P+\tfrac{1}{2}A - \tfrac{1}{2}F_\theta A + F_\theta B$$

$$= F_\theta\left(X-P-\tfrac{1}{2}A+B\right)+P+\tfrac{1}{2}A$$

$$= F_\theta\left([X+B]-(P+\tfrac{1}{2}A)\right)+P+\tfrac{1}{2}A$$

$$= F_\theta\left([X+B]-Q\right)+Q$$

$$= M_k\left(\mathcal{T}_B(X)\right)$$

where $k = \mathcal{T}_{A/2}(\ell)$, i.e. $k \parallel \ell$, k contains $Q = P + \tfrac{1}{2}A$.

We've proven:

<u>Prop</u> With above setup, $\mathcal{T}_u \circ M_\ell = \mathcal{G}(X)$ with glide B, mirror line $k = \ell$ transl'd by $\tfrac{1}{2}A$.

<u>You check:</u> $M_\ell \circ \mathcal{T}_u$ is also a glide refl'n.

We have (finally!) exhausted all possibilities, and have proven:

<u>Thm</u> The only possible isometries of $\mathbb{R}^2$ are:

- $\mathcal{I}(x)$, the identity.

  or'n preserving, involution, fixed pts $= \mathbb{R}^2$.
  comp'n of 0 or 2 refl'ns ($\mathcal{I} = M_\ell \circ M_\ell \ \forall \ell$)

- $M_\ell(x)$, refl'n across ℓ

  or'n reversing, involution, fixed pts $= \ell$.

- $R(x)$ rot'n by θ about C

  or'n preserving, not invol'n (unless $\theta = 0, \pi$), fixed pts $= \{c\}$ (unless $\theta = 0$). Comp'n of 2 refl'n.

  <u>Special rot'ns</u>: $R_{0,c}(x) = \mathcal{I}(x)$, $R_{\pi,c}(x) = \mathcal{E}_c(x) = 2C - X$ (invol'n)

  rot'ns by $\theta \notin \{0, \pi\}$ called <u>non-special</u>.

- $\mathcal{T}_u(x)$, transl'n by u

  or'n preserving, non invol'n, <u>no</u> fixed pts unless $u = 0$, which is <u>degenerate</u>: $\mathcal{T}_0(u) = \mathcal{I}(x)$. Comp'n of two refl'ns.

- $\mathcal{G}(x)$, glide refl'n, slide by u, refl'n across ℓ, $u \parallel \ell$.

  or'n <u>reversing</u>, not invol'n, <u>no</u> fixed pts (<u>unless $u = 0$</u>)
  comp'n of <u>three</u> refl'ns.

  degenerate glide refl'n