# Stratified categories, geometric fixed points and a generalized Arone-Ching theorem: talk notes

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Thanks to the organizers for inviting me to speak. Convention: everything in this talk will be homotopy invariant, so all of my categories will be $\infty$-categories, all of my colimits homotopy colimits, and so on. I'll begin by giving rapid-fire refreshers on both functor calculus and equivariant stable homotopy theory.

## 1 Functor calculus

All results, unless otherwise stated, are due to Goodwillie. Let **Sp** be the category of spectra and let $F : \mathbf{Sp} \to \mathbf{Sp}$ be a functor. All of our functors will be reduced: that is,

$$F(*) = *.$$

An $n$-cube of spectra will be called *strongly cocartesian* if it's constructed by writing down $n$ morphisms

$$X \to Y_1, X \to Y_2, \cdots, X \to Y_n$$

and then taking iterated pushouts. (Example with 3-cube.) $F$ will be called $n$-*excisive* if it takes strongly cocartesian $n$-cubes to cartesian $n$-cubes - cubes which are limit diagrams, or equivalently (since we're stable) colimit diagrams.

Any functor has a right universal $n$-excisive approximation $F \to P_n F$. $F$ will be called $n$-homogeneous if $F = P_n F$ and $P_{n-1} F = 0$.

**Proposition 1.1.** $F$ is $n$-homogeneous if and only if it's equivalent to a functor of the form

$$F(X) = (D \wedge X^{\wedge n})_{h\Sigma_n}$$

for some spectrum $D$ with $\Sigma_n$-action.

$D$ is supposed to look like a Taylor coefficient; this formula is a categorification of

$$f(x) = \frac{dx^n}{n!}.$$

For example, the fiber of $P_n F \to P_{n-1} F$ is $n$-homogeneous. The coefficient spectrum $D_n$ arising here is called the *nth derivative of* $F$. So we have a fiber sequence

$$(D_n \wedge (-)^{\wedge n})_{h\Sigma_n} \to P_n F \to P_{n-1} F.$$

In particular, suppose $F$ is 2-excisive. Then $P_1F$ is 1-homogeneous, and so we have a fiber sequence

$$(D_2 \wedge X \wedge X)_{h\Sigma_2} \to F(X) \to D_1 \wedge X.$$

The last general thing I want to say about calculus is this proposition - I'm not sure who to credit with this, but it's well-known.

**Proposition 1.2.** If $F$ is $n$-excisive, then it's determined by its restriction to the full subcategory of $\mathbf{Sp}$ spanned by the wedges of up to $n$ copies of the sphere spectrum $\mathbb{S}$.

This suggests that $n$-excisive functors might be somehow combinatorial at heart. We'll return to this.

# 2 Equivariant homotopy theory

Let $G$ be a finite group. The first realization in equivariant stable homotopy theory is that a $G$-spectrum is much more than a spectrum with a $G$-action. There are several models of what we call *genuine $G$-spectra*, but in this talk by a $G$-spectrum we'll mean a *spectral Mackey functor*, which I'll now explain.

Let $\mathbf{C}$ be a category with pullbacks. Then we can form the *effective Burnside category* of $\mathbf{C}$, denoted $A^{eff}(\mathbf{C})$, whose objects are those of $\mathbf{C}$ and in which morphisms are spans (draw). Compositions are defined by forming pullbacks.

If $\mathbf{C}$ is *disjunctive* - that is, it has finite coproducts and pullbacks distribute over them - then $A^{eff}(\mathbf{C})$ is *semiadditive* - that is, it has a zero object and finite direct sums, so that finite coproducts coincide with finite products and are given by the coproduct in $\mathbf{C}$.

**Definition 2.1.** A *Mackey functor* on $\mathbf{C}$ is a direct-sum-preserving functor from $A^{eff}(\mathbf{C})$ to $\mathbf{Sp}$.

**Definition 2.2.** A *$G$-spectrum* is a Mackey functor on the category $\mathcal{F}^G$ of finite $G$-sets.

What does this amount to? Let's restrict attention to the irreducible $G$-sets of the form $G/H$, where $H$ is a subgroup of $G$. Then a $G$-spectrum $E$ gives a spectrum $E^H$ - the "genuine fixed points" - for each $H$, which has an appropriate group action, and whenever $H \leq K$, we get a restriction map $E^K \to E^H$ and a transfer map $E^H \to E^K$.

This picture is adapted to dealing with the genuine fixed points of spectra, which are both a left and a right adjoint and which are compatible with $\Omega^\infty$ but not as easily related to $\Sigma^\infty$. There's another notion of fixed points - the *geometric fixed points* $E \mapsto \Phi^H E$ - which are a left adjoint only, but play well with the formation of suspension spectra.

Here's an important example of a $G$-spectrum. Let $A$ be a spectrum, and form $A^{\wedge G}$: the smash product of copies of $A$ indexed by $G$, with $G$ permuting the factors, also known as the "HHR norm". Making this into a genuine $G$-spectrum is surprisingly subtle - one can construct it at the point set level, and then proving homotopy invariance is a nightmare, but from the Mackey functor point of view it's not even how to start. This is because the genuine fixed point spectrum $(A^{\wedge G})^G$ is full of mysteries; for example, it's intimately connected to Witt vectors, which is something I'm still amazed by every day. I mean, all you did was take some fixed points.

On the other hand, we have

$$\Phi^G(A^{\wedge G}) \simeq A.$$

This raises the question: can we present a $G$-spectrum in terms of its geometric fixed point spectra?

# 3 The dictionary

Now suppose that $G = C_2$. Then there is a "norm cofibration sequence", due to Greenlees and May:

$$E_{hG} \to E^G \to \Phi^G E.$$

In particular, if $E$ is the norm $A \wedge A$, then this sequence takes the form

$$(A \wedge A)_{hC_2} \to (A \wedge A)^{C_2} \to \Phi^{C_2}(A \wedge A) \simeq A.$$

Strikingly, this sequence coincides with the Taylor tower for the 2-excisive functor $A \mapsto (A \wedge A)^{C_2}$. We see that the first and second derivatives of this functor are both $\mathbb{S}$. In fact, this is the motivating case for the following even more striking theorem:

**Theorem 3.1** (?)**.** There is an equivalence between the category of $C_2$-spectra and the category of 2-excisive functors which takes a $C_2$-spectrum $D$ to the functor

$$A \mapsto (A \wedge A \wedge X)^{C_2}.$$

This theorem establishes a dictionary between equivariance and calculus, which I'll deliberately write without reference to $C_2$ or the number 2:

| $G$-spectrum $E$ | $n$-excisive functor $F$ |
|---|---|
| Norm cofibration sequence and its generalizations | Taylor tower |
| Geometric fixed points | Derivatives |
| Genuine fixed points for full group $E^G$ | Value $F(\mathbb{S})$ |
| Presentation by geometric fixed points | Arone-Ching theorem |

Challenge: can we come up with a framework that incorporates both these phenomena?

# 4 Epiorbital categories

The answer is yes, because $n$-excisive functors are Mackey functors too. Before I say exactly how, I want to fix a nice class of indexing categories:

**Definition 4.1.** A category is *epiorbital* if

1. it's an essentially finite 1-category, i.e. it has finitely many isomorphism classes of objects and finite hom-sets;
2. all morphisms are epimorphisms;
3. pushouts and coequalizers exist.

The first two conditions imply that if $X$ and $Y$ are two objects and morphisms between them exist in each direction, then those morphisms are isomorphisms. So an epiorbital category should be thought of as a "poset with automorphisms".

The most important examples for the purposes of this talk are the orbit category $\mathbf{O}_G$ of a finite group $G$ (i.e. irreducible $G$-sets) and the category $\mathcal{F}_s^{\leq n}$ of finite sets of cardinality at most $n$ and surjections.

**Proposition 4.2.** If $\mathbf{C}$ is an epiorbital category then the category $\mathbf{C}^{\amalg}$ obtained by freely adjoining finite coproducts to $\mathbf{C}$ is disjunctive. (Remember that disjunctive categories are the ones whose categories of spans behave nicely.)

**Pretheorem 4.3** (G.)**.** The category of $n$-excisive functors $\mathbf{Sp} \to \mathbf{Sp}$ is equivalent to the category of Mackey functors on $(\mathcal{F}_s^{\leq n})^{\amalg}$. If $F \in \mathrm{Fun}^{n-\mathrm{exc}}(\mathbf{Sp}, \mathbf{Sp})$, the value of the corresponding Mackey functor on $S$ is the cross effect

$$cr_S F(\mathbb{S}, \mathbb{S}, \cdots, \mathbb{S}).$$

This has pretheorem status in that I know it's true and I know how to prove it, but I have to nail down some technicalities about semiadditive $\infty$-categories. Note that this is a topological version of an algebraic theorem by Baues, Dreckmann, Franjou and Pirashvili.

What's the analog of geometric fixed points for Mackey functors on an arbitrary epiorbital category? Let $\mathbf{C}$ be epiorbital and $X \in \mathbf{C}$. Let $\mathbf{C}^{\leq X}$ be the full subcategory of $\mathbf{C}$ spanned by the objects admitting a map from $X$; these are all different for nonisomorphic $X$. There's a functor

$$j_X : \mathbf{C}^{\amalg} \to (\mathbf{C}^{\leq X})^{\amalg}$$

given by setting all objects outside $\mathbf{C}^{\leq X}$ to the empty set; then the "geometric fixed points at $X$" are given by left Kan extension along $j_X$. This recovers ordinary geometric fixed points in the case of $G$-spectra and derivatives in the case of $n$-excisive functors.

## 5 The reconstruction theorem

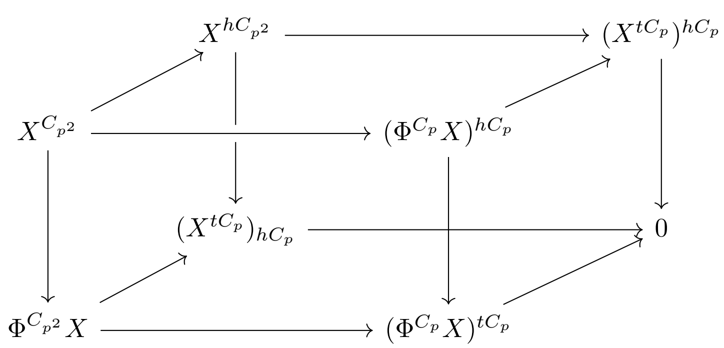
Now we'll investigate how to reconstruct Mackey functors on epiorbital categories from their geometric fixed point spectra. For motivation, let's first return to the case of $C_2$-spectra. We have a map of cofiber sequences

$$\begin{array}{ccccc} E_{hC_2} & \longrightarrow & E^{C_2} & \longrightarrow & \Phi^{C_2} E \\ \| & & \downarrow & & \downarrow \\ E_{hC_2} & \longrightarrow & E^{hC_2} & \longrightarrow & E^{tC_2} \end{array}$$

where the bottom cofiber sequence is the definition of the Tate spectrum $E^{tC_2}$. Thus the right hand square is a pullback square, and we learn that the glue needed to reconstruct $E^{C_2}$ from the geometric fixed point spectra of $E$ is the map from $\Phi^{C_2} E$ to $E^{tC_2}$. In previous work on this subject, Abram and Kriz have generalized this "fracture square" to $G$-spectra for abelian $G$, and Arone and Ching have given an analog for $n$-excisive functors. Our common generalization is as follows:

**Theorem 5.1** (G.)**.** Let $M$ be a Mackey functor on an epiorbital category $\mathbf{C}$. Then there is an explicit cartesian (Iso $\mathbf{C}$)-cube with limit $M$ whose other vertices are given by homotopy-theoretic constructions (homotopy orbits and fixed points, Tate spectra) on geometric fixed pont spectra of $M$.

OK, that's a tangle. Let's see it in action. We've seen the case where $\mathbf{C} = \mathbf{O}_{C_2}$, and $C_p$ for prime $p$ is exactly the same. Let's take $\mathbf{O}_{C_{p^2}}$ this time. For convenience, I'll evaluate the entire cube of Mackey functors on $G/G$.




The $(X^{tC_p})_{hC_p}$ entry requires a little computation. The deepest entry turns out to be 0, and it's not unusual for all entries at depth at least 3 to be 0.

## 6 Chromatic theory

Fix a prime $p$ and let $K(n)$ be the $n$th Morava $K$-theory. It's a deep theorem coming from I guess work of Greenlees-Sadofsky and Hovey-Sadofsky that Tate spectra all vanish in the the category of $K(n)$-local spectra. Nick Kuhn used this to show that Taylor towers all split in the $K(n)$-local category.

Here's a general theorem along those lines:

**Proposition 6.1** (G.)**.** Every entry of the cube of depth at least 2 contains a Tate construction, and so vanishes in the $K(n)$-local category. This gives a product decomposition of any Mackey functor valued in $K(n)$-local spectra. In the case of calculus, this gives Kuhn's splitting; for $G$-spectra, it gives a version of the tom Dieck splitting.

# 7 Stratified $\infty$-categories

Let's say a couple of words about the proof of the main theorem. In fact, the category $\mathbf{Mack}(\mathbf{C}^{\amalg})$ for an EOC $\mathbf{C}$ is an example of an abstract structure known as a *stratified stable $\infty$-category*:

**Definition 7.1.** Let $\mathcal{P}$ be a finite poset and let $\mathcal{I}_{\mathcal{P}}$ be the poset of intervals in $\mathcal{P}$, ordered by inclusion. A *stratification* of a stable $\infty$-category $\mathbf{D}$ along a finite poset $\mathcal{P}$ is an exact localization $\mathcal{L}_I$ for each interval $I \subseteq \mathcal{P}$ such that

1. for $I_1 \subseteq I_2$, $I_2$-local implies $I_1$-local, and
2. whenever $I$ and $J$ are intervals such that there are no $j \in J$ and $i \in I$ with $j > i$, the natural diagram

$$\begin{array}{ccc} \mathcal{L}_{I\cup J} & \longrightarrow & \mathcal{L}_I \\ \downarrow & & \downarrow \\ \mathcal{L}_J & \longrightarrow & \mathcal{L}_J\mathcal{L}_I \end{array}$$

   is a pullback diagram of functors.

Then there's a general reconstruction theorem that expresses any object of $\mathbf{D}$ as a limit of objects which belong to $\mathcal{L}_{\{p\}}\mathbf{D}$ for elements $p \in \mathcal{P}$. So your minimal intervals.

For $\mathbf{Mack}(\mathbf{C}^{\amalg})$, the poset $\mathcal{P}$ is the underlying poset of $\mathbf{C}$. For an interval $\mathcal{I}$ which is downwards-closed, the $\mathcal{I}$-local objects are those Mackey functors which are supported on the objects in $\mathcal{I}$, and for general intervals, it's something slightly more complicated that I can tell you about if you're interested.

There are other examples of stratified categories too: for instance, the category of $E_n$-local spectra is stratified by localizations at wedges of Morava K-theories, and our reconstruction theorem applies there too.