The Rogers-Ramanujan identities - a historical view

Oct. 5, 2018 Univ. of Minnesota combinatorics seminar

① History since 1894

② Refinements - joint work with K. O'Hara.

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} (1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1}$$

1st R-R identity

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} (1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}$$

2nd R-R identity

NOTATION $(A)_N = (A;q)_N = (1-A)\cdots(1-Aq^{N-1})$

$N = \infty$ IS OK

$$\begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Math Reviews

917 papers (94 by G. Andrews)

244 2010-2018

11 theses

2018 book by A. Sills

1913 Ramanujan's letter to Hardy

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\ddots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}$$

$$1 - \cfrac{e^{-\pi}}{1+\cfrac{e^{-2\pi}}{1-\cfrac{e^{-3\pi}}{1-\ddots}}} = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right) e^{\pi/5}$$

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}} = \frac{\prod_{n=0}^{\infty}(1-q^{5n+1})(1-q^{5n+4})}{\prod_{n=0}^{\infty}(1-q^{5n+2})(1-q^{5n+3})}$$

Ramanujan had no proof until 1917.

Rogers papers 1894-5 had RR identities proven.

Rogers had studied expansions of infinite products.

Today "q-Hermite" expansions.

Modern view of Rogers' proof (S. 2001)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-xt+t^2/2} \, e^{-x^2/2} \, dx = I(t) = e^{t^2}.$$

$$\frac{(q)_\infty}{2\pi} \int_{-\pi}^{\pi} (te^{-i\theta}, te^{i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_\infty \, d\theta = I_q(t)$$

$$= \sum_{N=0}^{\infty} \frac{q^{N^2-N} t^{2N}}{(q)_N}$$

This comes from q-Hermite orthogonality

Specializing t, Fourier orthogonality $\Rightarrow$ product side

$$1^2 + 2^2 = 5$$

Ramanujan's proof 1917

$$\sum_{n=0}^{\infty} \frac{q^{n^2} t^n}{(q)_n} = \frac{1}{(tq)_\infty} \sum_{n=0}^{\infty} (-t)^n t^{2n} q^{\frac{5}{2}n^2 + \frac{3n}{2}} (1 - tq^{2n}) \frac{(tq)_{n-1}}{(q)_n}$$

use Jacobi-Triple Product

$$\sum_{m=-\infty}^{\infty} q^{m^2} (-x)^m = (q^2;q^2)_\infty (qx;q^2)_\infty (q/x;q^2)_\infty$$

POSITIVE = BIGGER ALTERNATING

+

CANCELLATION (EXTRA)

MacMahon gave the combinatorial interpretation

THM # of partitions of n into parts with difference ≥ 2

equals

# of partitions of n into parts $\equiv 1, 4 \pmod 5$.

EX $n = 10$

$$\left\{\begin{array}{l} 10 \\ 9+1 \\ 8+2 \\ 7+3 \\ 6+4 \\ 6+3+1 \end{array}\right. \qquad \left\{\begin{array}{l} 9+1 \\ 6+1+1+1+1 \\ 4+4+1+1 \\ 6+4 \\ 4+1+1+1+1+1 \\ 1+1+1+1+1+1+1+1+1+1 \end{array}\right.$$

no 1's

$$\left\{\begin{array}{l} 10 \\ 8+2 \\ 7+3 \\ 6+4 \end{array}\right. \qquad \left\{\begin{array}{l} 8+2 \\ 7+3 \\ 3+3+2+2 \\ 2+2+2+2+2 \end{array}\right.$$

Schur 1917 also found and proved RR.

$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} q^{k^2} \begin{bmatrix} n+1-k \\ k \end{bmatrix}_q = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(5k+1)/2} \begin{bmatrix} n+1 \\ \left[\frac{n+1-5k}{2}\right] \end{bmatrix}_q$$

LHS has a simple combinatorial interpretation.

LHS = q-Fibonacci number

Golden ratio formula for F_n $\Rightarrow$

$$F_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k+1} 5^k$$

??

Other "explanations" of S

Watson's q-analogue of Whipple's transformation

$${}_4\phi_3\left(\quad\right) = (\quad)\,{}_8W_7\left(a; b, c, d, e, f\right) \qquad (1930's)$$

$b, c, d, e, f \to \infty$ gives Ramanujan's identity

Bailey defined "Bailey pairs" (1940's)

Slater found >100 R-R type identities (1951)

single sum = product with modular condition

$$\Pi(1-q^{12n-2})(1-q^{12n-10})(1-q^{12n}) = \Pi(1-q^n)\,\Sigma\,\frac{(-q;\ q^2,\ n)\,q^{n(n+2)}}{(q;\ 2n+1)}$$

I(18) $y,\ z\to\infty$. (50)

$$\Pi(1-q^{12n-4})(1-q^{12n-8})(1-q^{12n}) = \Pi(1-q^n)\,\Sigma\,\frac{(-q;\ q^2,\ n)\,q^{n(n+1)}}{(q;\ 2n+1)}$$

K(3) $y,\ z\to\infty$. (51)

$$= \Pi\,\frac{(1-q^{2n})}{(1+q^{2n-1})}\,\Sigma\,\frac{(-q;\ q^2,\ n)\,q^{n(2n-1)}}{(q^2;\ q^4,\ n)(q^2;\ q^2,\ n)}$$

C(5) $y=-q^{\frac{1}{2}},\ z\to\infty$. (52)

$$\Pi(1-q^{12n-5})(1-q^{12n-7})(1-q^{12n})$$

$$= \Pi(1-q^{4n})\,\Sigma\,\frac{(q;\ q^2,\ 2n)\,q^{4n^2}}{(q^4;\ q^4,\ 2n)}$$ L(2) $y,\ z\to\infty$. (53)

$$= \Pi(1-q^n)\left\{1+\sum_{n=1}^{\infty}\frac{(-q;\ q^2,\ n-1)(1+q^n)\,q^{n^2}}{(q;\ 2n)}\right\}$$ I(14) $y,\ z\to\infty$. (54)

or from (47) and (48) by addition.

$$\Pi(1-q^{12n-1})(1-q^{12n-11})(1-q^{12n}) = \Pi(1-q^{4n})\,\Sigma\,\frac{(q;\ q^2,\ 2n+1)\,q^{4n(n+1)}}{(q^4;\ q^4,\ 2n+1)}$$

M(3) $y,\ z\to\infty$. (55)

$$\Pi(1+q^{12n-1})(1+q^{12n-11})(1-q^{12n}) = \Pi(1-q^n)\,\Sigma\,\frac{(-q;\ n-1)\,q^{n^2-1}}{(q;\ q^2,\ n)(q;\ n)}$$

I(17) $y,\ z\to\infty$. (56)

$$= \Pi(1-q^{4n})\,\Sigma\,\frac{(-q;\ q^2,\ 2n+1)\,q^{4n(n+1)}}{(q^4;\ q^4,\ 2n+1)}$$

M(2) $y,\ z\to\infty$. (57)

$$\Pi(1+q^{12n-5})(1+q^{12n-7})(1-q^{12n}) = \Pi(1-q^n)\,\Sigma\,\frac{q^{n^2}(q^2;\ q^2,\ n-1)}{(q;\ q^2,\ n)(q;\ n)(q;\ n-1)}$$

I(9) $y,\ z\to\infty$. (58)

$$\Pi(1-q^{14n-2})(1-q^{14n-12})(1-q^{14n}) = \Pi(1-q^n)\,\Sigma\,\frac{q^{n(n+2)}}{(q;\ q^2,\ n+1)(q;\ n)}$$

C(4) $y,\ z\to\infty$. (59)

THM (Gordon, 1961)

# of partitions of n into parts $\not\equiv 0, \pm t \bmod (2k+1)$

equals

# of partitions of n $\lambda_1+\lambda_2+\cdots+\lambda_s$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$

$\lambda_s \geq \lambda_{s+k-1}+2$, $\lambda_{s-t+1} \geq 2$.

He had NO corresponding q-identity.

Andrews (1974)

THM $0 \le a \le k$ (Andrews-Gordon identities)

$$\frac{(q^{k+1-a}, q^{k+2+a}, q^{2k+3}; q^{2k+3})_\infty}{(q)_\infty} =$$

$$\sum_{n_1 \ge \cdots \ge n_k \ge 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2 + n_{k+1-a} + \cdots + n_k}}{(q)_{n_1 - n_2} (q)_{n_2 - n_3} \cdots (q)_{n_{k-1} - n_k} (q)_{n_k}}$$

Even moduli Bressoud 1980

Stembridge 1990 - symmetric functions

$P_\lambda(\vec{x};q)$ = Hall-Littlewood polynomial

$$P_\lambda(z, zq, \ldots, zq^{n-1}; q) = z^{|\lambda|} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}_q$$

THM
$$\sum_{\ell(\lambda)\le k} z^{|\lambda|} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}_q = (-z)_n \sum_{r=0}^{n} (-1)^r z^{(k+1)r} q^{(k+2)\binom{r}{2}} \cdot \begin{bmatrix} n \\ r \end{bmatrix}_q \frac{(1-z^2q^{2r-1})}{(z^2q^{r-1})_{n-1}}$$

uses P_λ generating function of Macdonald

Specialize $z_i \to q^{i-1}$, $n \to \infty$

THM

$$\sum_{\ell(\lambda)\le k} \frac{q^{2n(\lambda)+2|\lambda|}}{(q)_\lambda} = \frac{1}{(q)_\infty} \prod_{n\equiv 0,\pm 1 \bmod 2k+3} (1-q^n)$$

$$\sum_{\ell(\lambda)\le k} \frac{q^{2n(\lambda)+|\lambda|}}{(q)_\lambda} = \frac{1}{(q)_\infty} \prod_{n\equiv 0,\pm(k+1) \bmod 2k+3} (1-q^n)$$

$$(z)_\lambda = (z)_{\lambda_1-\lambda_2}(z)_{\lambda_2-\lambda_3}\cdots$$

Jouhet-Zeng 2005

Warnaar 2006

Griffin-Ono-Warnaar 2016

Lepowsky + Milne Wilson — Characters of affine Lie algebras (1978-1982)

$A_1^{(1)} = sl_2^{\wedge}$ V = highest weight module $h_0^* + 2h_1^*$

Macdonald identities give the principal specialization

$$\chi(V) = \frac{1}{(q;q^2)_\infty} \frac{1}{(q,q^4;q^5)_\infty}$$ PRODUCT SIDE

filtration gives the sum side.

Many papers in Vertex Operators + Rep Theory

McCoy-Berkovich-Schilling-Warnaar (see ICM 1997)

Statistical Physics and Representation Theory

Minimal models in conformal field theory

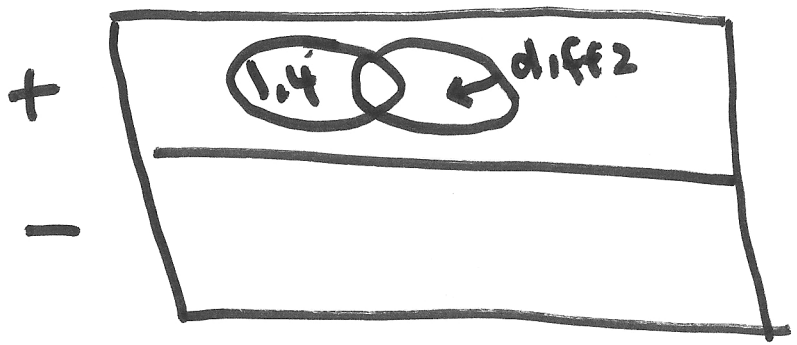
$$\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left(q^{j(10j+1+2a)}-q^{(2j+1)(5j+2-a)}\right)$$

$$=\sum_{n=0}^{\infty}\frac{q^{n(n+a)}}{(q)_n}$$

LHS = Bosonic RHS = Fermionic

Lovejoy 2018

Garsia-Milne: Bijection via Involution Principle 1981.



φ_1, φ_2 involutions, sign-reversing

$FP(\varphi_1)$ = partitions parts 1, 4 mod 5

$FP(\varphi_2)$ = partitions difference ≥ 2

They followed a proof of Schur, who had the most difficult involution.

NO DIRECT BIJECTION EXISTS TODAY

Zeilberger-Paule : Computer algebra (1990–)

Ekhad-Tre 1990

$$\sum_{k=0}^{n} q^{k^2} \frac{1}{(q)_k (q)_{n-k}} = \sum_{k=-n}^{n} (-1)^k \frac{q^{(5k^2-k)/2}}{(q)_{n-k} (q)_{n+k}}$$

LHS = 2nd order recurrence in n

RHS = 5th order " " " factor is LHS.

Paule simplified this (inserting $1+q^k$ on RHS) and "naturally" produced such identities.

Sills (2002–) all 130 Slater identities

Paule-Radu (2012–)

Fulman - random matrices over $GF(q)$ (2000-2002)

THM $\lim_{n\to\infty} P(A \text{ } n\times n \text{ matrix in } GF(q) \text{ is semisimple})$

$$= \prod_{n\equiv 0,\pm 2 \bmod 5} \left(1-\frac{1}{q^n}\right).$$

THM Same limit $A \in GL_n(q)$

$$= \prod_{r\equiv 0,\pm 2 \bmod 5} \frac{\left(1-\frac{1}{q^{rn}}\right)}{\left(1-\frac{1}{q^r}\right)}$$

Markov chain proof of RR

sum side : identities + measure

product side : transition matrix diagonalizable

Lewis-Reiner-S. : Invariants of $GL_n(\mathbb{F}_q)$ (2017)

CONJ $S = \mathbb{F}_q[x_1, \ldots, x_n]$ $\quad I_m = \langle x_1^{q^m}, \ldots, x_n^{q^m} \rangle$ ideal

$$\mathrm{Hilb}\left(\left(S/I_m\right)^{GL_n(\mathbb{F}_q)}, t \right)$$

$$= C_{nm}(t) = \sum_{k=0}^{\min(m,n)} t^{(n-k)(q^m - q^k)} \begin{bmatrix} m \\ k \end{bmatrix}_{q,t}$$

$$\lim_{n\to\infty} \lim_{q\to 1} C_{nn}\left(t^{1/q-1}\right) = \frac{1}{(t, t^4, t^5)_\infty}$$

??

Baxter: Hard hexagon model (rediscovered! 1981)

Nice reminiscences paper of 2015 J. Phys. A

Colored Jones polynomial and knots

Hikami 2003 Garoufalidis-Li-Zagier 2015

Keilthy-Osburn 2016 Lovejoy 2017

Witten-Reshetikhin-Turaev invariants

New R-R with difference conditions

Kanade-Russell (2014 - ...)

Bringmann-Jennings-Schaffer-Mahlburg 2018

Garrett-Ismail-S. : m-versions 1999

THM
$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q)_n} = \frac{a_m(q)}{(q,q^4;q^5)_\infty} + \frac{b_m(q)}{(q^2,q^3;q^5)_\infty}$$

$a_m(q), b_m(q) \sim$ Schur polynomials

Quintic transformation

THM
$$\sum_{n=0}^{\infty} \frac{q^{n^2}(tq)^{2n}}{(q)_{2n}} = \frac{(t^4q^1, t^2q^5, t^6q^6; q^5)_\infty}{(t^2q^3)_\infty} \times {}_3\phi_2\left(\begin{matrix} t^2q^2, t^2q^3, t^2q^5 \\ t^4q^9, t^4q^5 \end{matrix} \middle| q^5; t^2q^5\right)$$

no JTP mean?

New refinements of RR mod 5 (O'Hara-S. 2014, 2018)

+ other partition theorems

Marking a part

Marking a sum of parts

Variation: Shifting parts

(1) Marking a part $\frac{1}{1-q^9} \rightarrow \frac{1}{1-xq^9}$

(2) Shifting a part $\frac{1}{1-q^9} \rightarrow \frac{1}{1-q^{20}}$

(3) Marking a sum of parts : generalizes the multiplicity of a single part

44111 appears in 6664444411111111

two times

Refinements of

1. Euler's Odd = Distinct theorem
2. Rogers-Ramanujan identities
3. Andrews-Gordon identities
4. Göllnitz-Gordon identities

Both analytic and combinatorial

Theorem Let $M \equiv 1$ or $4 \bmod 5$.

partitions of n into parts $\equiv 1, 4 \bmod 5$ with k M's

$=$ # partitions of n with difference ≥ 2, λ,

(1) λ has one part, $\lfloor n/M \rfloor = k$

(2) λ has ≥ 2 parts, $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$.

Theorem Let $M \equiv 1$ or $4 \bmod 5$.
Let $N > M$, $N \not\equiv 1, 4 \bmod 5$.

\# of partitions of n into parts $\equiv 1, 4 \bmod 5$
except M, or N

$=$ \# of partitions of n with difference ≥ 2, λ,

① λ has a single part $\equiv 0, 1, \ldots, M-1 \bmod N$

② λ has at least two parts,
$\lambda_1 - \lambda_2 - 2 = 0, 1, \ldots, M-1 \bmod N$.

Theorem Let $A = \{2, 3, 6, 9, 11, 14, 16, \ldots\}$

partitions of n into parts from A

$=$ # partitions of n with difference $\geq 3, \lambda,$

① if λ has a single part, then $n \not\equiv 1 \mod 3$

② if λ has at least two parts, then

$$\lambda_1 - \lambda_2 - 2 \not\equiv 1 \mod 3.$$

DEFN $E_\lambda(\mu)$ = # times λ appears in μ

EXAMPLE $E_{441}(44444111111) = 2.$

PROP For any part sizes A, λ a partition with parts from A, $\lambda \vdash M$

$$\frac{1-q^M}{1-xq^M} \prod_{a \in A} \frac{1}{1-q^a} = \sum_{\substack{\mu \text{ parts} \\ \text{in } A}} x^{E_\lambda(\mu)} q^{\|\mu\|}.$$

FACT # partitions of n into parts $\equiv 1,4 \bmod 5$

$\geq$ # partitions of n into parts $\equiv 2,3 \bmod 5$.

DEFN Fix θ partition with parts from $\mathcal{A}$.

$$E_\theta^{\mathcal{A}}(n,k) = \#\text{ partitions of } n \text{ into parts from } \mathcal{A} \text{ containing } \theta \text{ } k \text{ times.}$$

THM $\mathcal{A} = \{1,4,6,9,\ldots\}$, $\mathcal{B} = \{2,3,7,8,\ldots\}$

$\lambda \vdash M$ from $\mathcal{A}$ $\quad$ $\theta \vdash M$ from $\mathcal{B}$

$$E_\theta^{\mathcal{B}}(n,k) \leq E_\lambda^{\mathcal{A}}(n,k) \quad \text{if } M \geq 3.$$

THM Let M be odd, $M \geq 1$.

partitions of n into odd parts with k M's

$=$ # partitions of n into distinct parts, λ,

① if λ has one part, $\lfloor n/M \rfloor = k$

② if λ has at least two parts,

$$\lfloor (\lambda_1 - \lambda_2 - 1)/M \rfloor = k.$$

NOTE: M-fishhook bijection exists

An Andrews-Gordon version uses Durfee dissections (George, 1979), (k,a)-admissible partitions

THM Let $0 \le a \le k$. $M \neq 0, \pm(k+1-a) \bmod 2k+3$

partitions of n into parts not congruent to $0, \pm(k+1-a) \bmod 2k+3$

with exactly j M's,

$=$ # $(k+1, k+1-a)$-admissible partitions of n such that there are $r \le k$ Durfee rectangles

① if $r=1$, 1×1 $0 \le a < k$, then λ is a single part of size $M_j, M_{j+1}, \ldots, M_{j+M-1}$

② if $r=1$, $a=k$, $\lambda=(\lambda_1,1)$ has size $M_j, M_{j+2}, \ldots M_{j+M-1}$, or M_{j+M+1}

③ if $r\ge 2$, $n_1=1$, then $\left\lfloor (\lambda_r-n_1)/M \right\rfloor = j$

④ if $n_1\ge 2$, then $\left\lfloor (\lambda_r-\lambda_2)/M \right\rfloor = j$

$$\frac{1-q^M}{1-xq^M}\,\frac{(q^{k+1-a},\,q^{k+2+a},\,q^{2k+3};q)_\infty}{(q)_\infty} =$$

$$1 + A + \sum_{n_1=2}^{\infty} \frac{q^{n_1^2+B}}{(q^2;q)_{n_1-1}} \frac{[M]_q}{1-xq^M}$$

$$+ \sum_{r=2}^{k} \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{q^{n_1^2+\cdots+n_r^2+n_{k+1-a}+\cdots+n_r}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q^2;q)_{n_r-1}}$$

$$\times \frac{[M]_q}{1-xq^M}$$

A, B depend upon a, M.

Remarks ① Fractional multiplicity?

② $q \geq 1$ integer

$$\prod_{n=0}^{\infty} \frac{1}{1 - t^{[2n+1]_q}} = 1 + \sum_{m=1}^{\infty} t^{[m]_q} \frac{1 - t^{q^m [m]_q}}{1 - t^{[m]_q}} \times \prod_{k=1}^{m-1} \frac{1 - t^{(q^k+1)[k]_q}}{1 - t^{[k]_q}}$$

partitions of n into q-odd parts

= # partitions of n into parts $[m]_q$ of multiplicity $\leq q^m$.

? mark parts ?

(3) mark more than one part ?

RR: $\{M\}$, $\{2,M\}$, $\{2,3,7,8\}$ (2015)

2,3,7:

$$\frac{1}{(1-tq^2)(1-wq^3)(1-vq^7)}\left[1+\sum_{k=1}^{7}\frac{q^{k^2+k}}{(q;q)_k}\right]$$

$$= \frac{1}{(1-q^2)(1-q^3)(1-q^7)} \times$$

$$1 + \frac{q^2(t+wq)}{1-tq^2} + q^6\frac{(w^2+vq+q^2)}{(1-tq^2)(1-wq^3)}$$

$$+ q^{12}\left(\frac{1+q+v^2q^2+vq^3+q^4+q^5+q^6}{(1-tq^2)(1-wq^3)(1-vq^7)}\right)$$

$$+ q^{20}\left(\frac{1+q+q^2+q^3+q^4+q^5+q^6}{(1-tq^2)(1-wq^3)(1-q^4)(1-vq^7)}\right)$$

$$+ \frac{q^{30}\left(1+q+q^2+q^3+q^4+q^5+q^6\right)}{(1-tq^2)(1-wq^3)(1-q^4)(1-q^5)(1-vq^7)}$$

$$+ \frac{q^{42}\left(1+q+q^2+q^3+q^4+q^5+q^6\right)}{(1-tq^2)(1-wq^3)(1-q^4)(1-q^5)(1-q^6)(1-vq^7)}$$

? Hilbert series, polytopes ?

Thank you!