GENERATING FUNCTIONS AND ORTHOGONALITY

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Abstract. Askey used an idea of Hermite, the polynomiality of an integral, to prove Jacobi's generating function for Jacobi polynomials. The same idea is used for generating functions of the Al-Salam-Chihara, continuous dual q-Hahn, Wilson, and Askey-Wilson polynomials.

1. Introduction

Dick Askey was a master of the mathematical literature, including historical documents. His marvelous short paper [1] on the Jacobi polynomial generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}T^{-1}(1-t+T)^{-\alpha}(1+t+T)^{-\beta}, \quad T=(1-2xt+t^2)^{1/2}$$

is a good example of this. Hermite had a proof of the Legendre polynomial orthogonality, using only polynomiality of an integral. Hermite's proof appeared in the letters between Stieltjes and Hermite. Askey showed that Hermite's idea could be applied to Jacobi polynomials to obtain the above generating function. Practically no calculations were necessary, just a change of variable in an integral. The purpose of this paper is extend Hermite's idea to more general polynomials in the Askey scheme.

Let's review Hermite's idea for a general orthogonal polynomial measure $d\mu(x)$. Suppose that a set of polynomials $p_n(x)$, degree$(p_n)=n$, is given by a generating function

$$G(x,t)=\sum_{n=0}^{\infty} a_n p_n(x)t^n.$$

Let k be a non-negative integer and put

$$I_k=\int_{-\infty}^{\infty} G(x,t)x^k d\mu(x).$$

We see that if $p_n(x)$ are the orthogonal polynomials for $d\mu(x)$, then I_k is a polynomial in t of degree k. Conversely if I_k is a polynomial in t of degree k, for all $k \geq 0$, then by uniqueness of orthogonal polynomials, $p_n(x)$ must be a multiple of the monic orthogonal polynomial for $d\mu(x)$, with generating function $G(x,t)$. Thus we see that polynomiality of an integral involving a generating function is equivalent to orthogonality of the polynomials. Askey established polynomiality for the integral for Jacobi polynomials, thus proving the Jacobi generating function.

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For the normalization in the generating function, one can either equate coefficients of the highest power of x, or specialize x. In §2-5 where we carry out this idea on four sets of polynomials, we do not show these normalization details, they are routine.

In one of our examples we need a slight extension of Hermite's idea. Suppose the generating function (as a formal power series in t) is

$$G(x,t)=\sum_{n=0}^{\infty} p_n(x)A_n(t)t^n,$$

where $A_n(t)$ is a formal power series in t with constant term 1. Let's compute again

$$I_k=\int_{-\infty}^{\infty} G(x,t)b_k(x)d\mu(x),$$

for some fixed polynomial $b_k(x)$ of degree k. This time if $p_n(x)$ are the orthogonal polynomials for $d\mu(x)$, we obtain a linear combination of $\{A_0(t), tA_1(t), \cdots, t^kA_k(t)\}$. So to prove orthogonality, we would need that I_k is a linear combination of these $k+1$ series. We use this method for the continuous dual q-Hahn polynomials in §3.

In three of our four examples the measure $d\mu(x)$ is a specialized Askey-Wilson measure, the remaining example is the $q=1$ case of Wilson. The Askey-Wilson integral is (here $x=\cos(\theta)$)

$$I_q(a,b,c,d)=\int_0^{\pi} w(\theta,a,b,c,d|q)d\theta=\frac{(ab,ac,ad,bc,bd,cd;q)_\infty}{(abcd;q)_\infty}. \tag{1.1}$$

where

$$w(\theta,a,b,c,d|q)=\frac{(q,e^{2i\theta},e^{-2i\theta};q)_\infty}{2\pi(ae^{-i\theta},ae^{i\theta},be^{-i\theta},be^{i\theta},ce^{-i\theta},ce^{i\theta},de^{-i\theta},de^{i\theta};q)_\infty}.$$

We will use the notation found in [4], and consider all power series in t as formal power series. Conditions on parameters for convergence of the integrals will be ignored, as these integrals could be replaced by formal linear functionals.

2. Al-Salam-Chihara polynomials

The Al-Salam-Chihara polynomials

$$Q_n(x;a,b|q)=\frac{(ab;q)_n}{a^n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},ae^{-i\theta},ae^{i\theta}\\ ab,0\end{matrix}\;\middle|\;q;q\right)$$

are degree n polynomials in $x=\cos(\theta)$. They also are orthogonal, as functions of θ with respect to $w(\theta,a,b,0,0|q)$ on $[0,\pi]$.

Their generating function is

$$GF_Q(a,b,t,q)=\sum_{n=0}^{\infty}\frac{Q_n(x;a,b|q)}{(q;q)_n}t^n=\frac{(at,bt;q)_\infty}{(te^{-i\theta},te^{i\theta};q)_\infty} \tag{2.1}$$

To prove (2.1) from Hermite's idea, we take the polynomial of degree k in $x=\cos(\theta)$ to be $(be^{-i\theta}, be^{i\theta}; q)_k$. We must show that

$$I_k = \int_0^\pi GF_Q(a,b,t,q)w(\theta,a,b,0,0|q)(be^{-i\theta}, be^{i\theta}; q)_k d\theta$$

is a polynomial in t of degree k. However it is easy to see that I_k may be evaluated by (1.1)

$$I_k = (at,bt;q)_\infty I_q(a,bq^k,t,0) = \frac{(at,bt;q)_\infty}{(at,btq^k,abq^k;q)_\infty} = \frac{(bt;q)_k}{(abq^k;q)_\infty},$$

which is a polynomial in t of degree k.

3. Continuous dual q-Hahn polynomials

The continuous dual q-Hahn polynomials $p_n(x;a,b,c|q)$ are the $d=0$ special cases of the Askey-Wilson polynomials, thus have weight $w(\theta,a,b,c,0|q)$ on $[0,\pi]$,

$$p_n(x;a,b,c|q) = \frac{(ab,ac;q)_n}{a^n} {}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{-i\theta}, ae^{i\theta} \\ ab, ac \end{matrix} \mid q;q\right).$$

They have the generating function (see [2] or [6, Th. 2.9])

$$GF_p(a,b,c,t,q) = \sum_{n=0}^\infty \frac{p_n(x;a,b,c|q)}{(q,abct;q)_n} t^n = \frac{1}{(abct;q)_\infty} \frac{(at,bt,ct;q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty}. \tag{3.1}$$

Because the left side in (3.1) is an expansion in terms of $t^n A_n(t) = t^n/(abct;q)_n$, we will evaluate the integral I_k, and then show it is a linear combination of

$$\{A_0(t), tA_1(t), \cdots, t^k A_k(t)\}.$$

To prove (3.1) from Hermite's idea, we take the polynomial of degree k in $x=\cos(\theta)$ to be $(ae^{-i\theta}, ae^{i\theta}; q)_k$. We must find

$$I_k = \int_0^\pi GF_p(a,b,c,t,q)w(\theta,a,b,c,0|q)(ae^{-i\theta}, ae^{i\theta}; q)_k d\theta$$

However it is easy to see that I_k may be evaluated by (1.1)

$$I_k = \frac{(at,bt,ct;q)_\infty}{(abct;q)_\infty} I_q(aq^k,b,c,t) = \frac{(at,bt,ct,abctq^k;q)_\infty}{(abct,abq^k,acq^k,atq^k,bc,bt,ct;q)_\infty} = \frac{1}{(abq^k,acq^k,bc;q)_\infty}\frac{(at;q)_k}{(abct;q)_k},$$

However because [4, (II.5)]

$$\frac{(at;q)_k}{(abct;q)_k} = {}_2\phi_1\left(\begin{matrix} q^{-k}, bc \\ abct \end{matrix} \mid q; atq^k\right),$$

as a function t, I_k is a linear combination of $\{1, t/(abct;q)_1, \cdots t^k/(abct;q)_k\}$.

4. Wilson polynomials

The Wilson polynomials in x^2 are defined by

$$W_n(x^2;a,b,c,d)=(a+b)_n(a+c)_n(a+d)_n\,{}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a-ix,a+ix\\ a+b,a+c,a+d\end{matrix};1\right)$$

whose weight function is

$$v(x;a,b,c,d)=\frac{1}{2\pi}\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2$$

(4.1)
$$J(a,b,c,d)=\int_{-\infty}^{\infty}v(x;a,b,c,d)dx=\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}.$$

The generating function is [5], [3, (71)]
(4.2)
$$\begin{aligned}GF_W(a,b,c,d,t)=&\sum_{n=0}^{\infty}\frac{(a+b+c+d-1)_n}{n!(a+b,a+c,a+d)_n}W_n(x^2;a,b,c,d)t^n\\ =&(1-t)^{1-a-b-c-d}{}_4F_3\left(\begin{matrix}(a+b+c+d-1)/2,(a+b+c+d)/2,a-ix,a+ix\\ a+b,a+c,a+d\end{matrix};\frac{-4t}{(1-t)^2}\right).\end{aligned}$$

This time we take the polynomial in x^2 of degree k to be $(b-ix,b+ix)_k$ and find

$$I_k=\int_{-\infty}^{\infty}v(x;a,b,c,d)GF_W(a,b,c,d,t)(b-ix,b+ix)_k dx.$$

Integrating the generating function term by term, using (4.1) to find $J(a+n,b+k,c,d)$ as the n^{th} term, we see that

$$I_k=H(1-t)^{1-a-b-c-d}{}_3F_2\left(\begin{matrix}(a+b+c+d-1)/2,(a+b+c+d)/2,a+b+k\\ a+b,a+b+c+d+k\end{matrix};\frac{-4t}{(1-t)^2}\right)$$

where H is a constant which is independent of t. This may be shown to be a polynomial in t of degree k using the well-poised ${}_3F_2$ quadratic transformation.

Proposition 4.1. *As formal power series in t,*

$${}_3F_2\left(\begin{matrix}A,B,C\\ 1+A-B,1+A-C\end{matrix};t\right)=(1-t)^{-A}{}_3F_2\left(\begin{matrix}A/2,(A+1)/2,1+A-B-C\\ 1+A-B,1+A-C\end{matrix};\frac{-4t}{(1-t)^2}\right).$$

We choose $A=a+b+c+d-1$, $B=c+d$ and $C=-k$ to establish polynomiality in t of I_k.

5. Askey-Wilson polynomials

The Askey-Wilson polynomials are

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{a^n} {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{-i\theta}, ae^{i\theta} \\ ab, ac, ad \end{matrix} \mid q;q\right), \quad x=\cos(\theta),$$

with weight function $w(\theta;a,b,c,d|q)$.

The Askey-Wilson case is carried out the same way.

Proposition 5.1. *As formal power series in t, an Askey-Wilson generating function is*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(abcd/q;q)_n}{(q,ab,ac,ad;q)_n}p_n(x;a,b,c,d|q)t^n \\ &=\frac{(tbcd/q;q)_\infty}{(t/a;q)_\infty}{}_6\phi_5\left(\begin{matrix}\sqrt{abcd/q}, \sqrt{abcd}, -\sqrt{abcd/q}, -\sqrt{abcd}, ae^{-i\theta}, ae^{i\theta} \\ ab, ac, ad, tbcd/q, aq/t\end{matrix} \mid q;q\right).\end{aligned}$$

Remark 5.2. *One must consider Proposition 5.1 in the formal power ring in t. Here the denominator factor $1/(aq/t;q)_n$ in the ${}_6\phi_5$ is $t^n/(t/aq;q)_n(-qa)^n q^{\binom{n}{2}}$, which does have a formal power series expansion in t.*

Proof. We compute I_k as before, choosing $(be^{-i\theta}, be^{i\theta};q)_k$ as the polynomial of degree k. Using (1.1) we find that

$$I_k = HH\frac{(tbcd/q;q)_\infty}{(t/a;q)_\infty}{}_5\phi_4\left(\begin{matrix}\sqrt{abcd/q}, \sqrt{abcd}, -\sqrt{abcd/q}, -\sqrt{abcd}, abq^k \\ ab, abcdq^k, tbcd/q, aq/t\end{matrix} \mid q;q\right)$$

where HH is a constant independent of t. The terminating well-poised ${}_3\phi_2$ quadratic transformation [4, (III.14)] with

$$z=t/a, \quad A=abcd/q, \quad B=q^{-k}, \quad C=cd$$

proves that I_k is a polynomial in t of degree k. $\square$

An analytic version of Proposition 5.1 was given by Rahman [7, (4.9)], [3, Th. 3.2].

6. Concluding remarks

The generating functions in (4.1) and Proposition 5.1 may be found by power series expansions. Nonetheless, the importance of the well-poised ${}_3F_2$ and ${}_3\phi_2$ quadratic transformations is emphasized here. They were previously used with Hermite's idea in [8].

References

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