

RATIOS OF JACKSON'S q -BESSEL FUNCTIONS AND q -LOMMEL POLYNOMIALS

JANG SOO KIM AND DENNIS STANTON

ABSTRACT. In 1993 Delest and Fédou showed that a generating function for connected skew shapes is given as a ratio $J_{\nu+1}/J_\nu$ of the Jackson's third q -Bessel functions when a parameter ν is zero. They conjectured that when ν is a nonnegative integer the coefficients of the generating function are rational functions whose numerator and denominator are polynomials in q with nonnegative integer coefficients, which is a q -analog of Kishore's 1963 result on Bessel functions. The first main result of this paper is a proof of the conjecture of Delest and Fédou. The second main result is a refinement of the result of Delest and Fédou: a generating function for connected skew shapes with bounded diagonals is given as a ratio of q -Lommel polynomials introduced by Koelink and Swarttouw. It is also shown that the ratio $J_{\nu+1}/J_\nu$ has two different continued fraction expressions, which give respectively a generating function for moments of orthogonal polynomials of type R_I and a generating function for moments of usual orthogonal polynomials. Orthogonal polynomial techniques due to Flajolet and Viennot are used.

CONTENTS

1. Introduction	1
2. The q -Kishore theorem for Jackson's third q -Bessel function	6
3. The q -Kishore theorem for Jackson's first q -Bessel function	11
4. Orthogonal polynomials of type R_I and continued fractions	13
5. Ratios of q -Lommel polynomials	15
6. q -Nörlund and Heine continued fractions	19
7. Open problems	23
Acknowledgments	24
References	24

1. INTRODUCTION

The main objects of study in this paper are ratios of q -Bessel functions and q -Lommel polynomials, and their combinatorial properties. Jackson defined three q -Bessel functions. Delest and Fédou gave a combinatorial interpretation for a certain ratio of Jackson's third q -Bessel functions. We will extend their result and prove their conjecture in this case. We also consider the other two Jackson's q -Bessel functions.

The *Bessel functions* $J_\nu(x)$ are defined by

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \sum_{n \geq 0} \frac{(-z^2/4)^n}{n!(\nu+1)_n}.$$

Date: September 7, 2020.

It is well known [29, p. 54] that the Bessel functions generalize both sine and cosine functions:

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin z}{z^{1/2}}, \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi}} \cdot \frac{\cos z}{z^{1/2}}.$$

Therefore their ratio gives the tangent function:

$$(1.1) \quad \frac{J_{1/2}(z)}{J_{-1/2}(z)} = \tan z.$$

A motivating example of this paper is the following result of Kishore [20], which was observed by Lehmer [22] in 1945. We consider the ratio $J_{\nu+1}(z)/J_\nu(z)$ is a formal power series in z .

Theorem 1.1 (Kishore, [20]). *We have*

$$(1.2) \quad \frac{J_{\nu+1}(z)}{J_\nu(z)} = \sum_{n=1}^{\infty} \frac{N_{n,\nu}}{D_{n,\nu}} \left(\frac{z}{2}\right)^{2n-1},$$

where

$$D_{n,\nu} = \prod_{k=1}^n (k + \nu)^{\lfloor n/k \rfloor},$$

and $N_{n,\nu}$ is a polynomial in ν with nonnegative integer coefficients.

Throughout this paper we use the standard notation for q -series:

$$(a; q)_n = (1-a)(1-aq) \dots (1-aq^{n-1}), \quad (a; q)_\infty = (1-a)(1-aq) \dots,$$

$${}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) = \sum_{n \geq 0} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{s+1-r} z^n.$$

We also define $[\nu]_q = (1 - q^\nu)/(1 - q)$. Note that if n is a nonnegative integer, then $[n]_q = 1 + q + \dots + q^{n-1}$.

In 1993 Delest and Fédou [6, Conjecture 9] conjectured a q -analog of Kishore's theorem using Jackson's third q -Bessel functions, also known as Hahn–Exton q -Bessel functions [21].

Definition 1.2. The *Jackson's third q -Bessel functions* $J_\nu(z; q)$ are defined by

$$J_\nu(z; q) = \frac{(q^{\nu+1}; q)_\infty z^\nu}{(q; q)_\infty} {}_1\phi_1(0; q^{\nu+1}; q, qz^2).$$

From the definition it follows that

$$(1.3) \quad \lim_{q \rightarrow 1^-} J_\nu(z(1-q); q) = J_\nu(2z).$$

The first main result of this paper is the following theorem, which is slightly more general than the conjecture of Delest and Fédou [6, Conjecture 9]. See Remark 2.3 for the original statement of their conjecture.

Theorem 1.3. *For any ν (not a negative integer) we have*

$$\frac{J_{\nu+1}((1-q)z; q^{-1})}{J_\nu((1-q)z; q^{-1})} = - \sum_{n=1}^{\infty} \frac{N_{n,\nu}(q)}{D_{n,\nu}(q)} q^{(\nu+1)n} z^{2n-1},$$

where each $N_{n,\nu}(q)$ is a polynomial in q, q^ν and $[\nu]_q$ with nonnegative integer coefficients and

$$D_{n,\nu}(q) = \prod_{k=1}^n [k + \nu]_q^{\lfloor n/k \rfloor}.$$

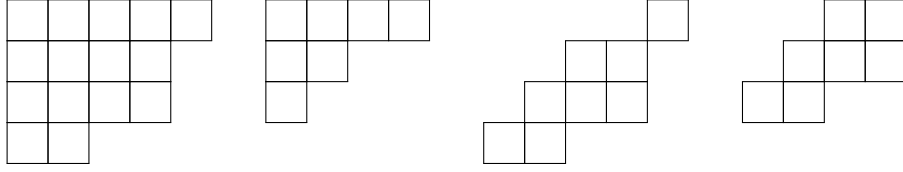


FIGURE 1. From left to right are shown the Young diagrams of the partitions $(5, 4, 4, 2)$ and $(4, 2, 1)$ and the skew shapes $(5, 4, 4, 2)/(4, 2, 1)$ and $(4, 4, 2)/(2, 1)$. The third diagram is not connected and the fourth diagram is connected. If α is the fourth diagram, then $\text{col}(\alpha) = 4$, $\text{row}(\alpha) = 3$, and $\text{area}(\alpha) = 7$.

Note that by (1.3) the $q \rightarrow 1^+$ limit of Theorem 1.3 with z replaced by $-q^{-1}z/2$ recovers the Kishore theorem, Theorem 1.1. Our method of proof also works for the Jackson's third q -Bessel functions with the usual q base instead of q^{-1} and for Jackson's q -Bessel functions, see Theorems 2.6 and 3.2.

Our second main result concerns combinatorial properties of the ratio of Jackson's third q -Bessel functions. We first introduce necessary definitions.

Definition 1.4. A *partition* is a sequence $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of positive integers with

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n.$$

The *Young diagram* of a partition σ is a left-justified array of squares in which the i th row has σ_i squares.

If σ and ρ are partitions such that the Young diagram of ρ is contained in that of σ , the *skew shape* σ/ρ is defined to be the set-theoretic difference of their Young diagrams. Two skew shapes are considered as the same skew shape if one is obtained from the other by translation.

A skew shape σ/ρ is *connected* if for any two squares u and v in σ/ρ there is a sequence u_0, u_1, \dots, u_k of squares in σ/ρ such that $u_0 = u$, $u_k = v$ and for each $1 \leq i \leq k$ the squares u_i and u_{i-1} share an edge.

Denote by CS the set of all nonempty connected skew shapes. For $\alpha \in \text{CS}$, let $\text{col}(\alpha)$ be the number of nonempty columns in α , $\text{row}(\alpha)$ the number of nonempty rows in α , and $\text{area}(\alpha)$ the number of squares in α . See Figure 1.

Two skew shapes are considered to be equal if one is obtained from the other by translation. For example, the skew shape $(4, 4, 2)/(2, 1)$ in Figure 1 is the same as $(5, 4, 4, 2)/(5, 2, 1)$ or $(5, 5, 3)/(3, 2, 1)$.

Delest and Fédou [6] showed that a generating function for connected skew shapes can be written as a ratio of Jackson's third q -Bessel functions. Bousquet-Mélou and Viennot [4] generalized their result by adding one more parameter.

Theorem 1.5 ([6] for $\nu = 0$ and [4] for general ν). *We have*

$$\sum_{\alpha \in \text{CS}} (q^\nu z^2)^{\text{col}(\alpha)} (q^\nu)^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = -q^\nu z \frac{J_{\nu+1}(z; q^{-1})}{J_\nu(z; q^{-1})}.$$

In fact Delest and Fédou [6] (for $\nu = 0$), and Bousquet-Mélou and Viennot [4] state their results in the following equivalent form:

$$\sum_{\alpha \in \text{CS}} x^{\text{col}(\alpha)} y^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = \frac{qxy}{1-xy} \cdot \frac{{}_1\phi_1(0; q^2y; q, q^2x)}{{}_1\phi_1(0; qy; q, qx)}.$$

Bousquet-Mélou and Viennot [4] also showed that

$$(1.4) \quad \sum_{\alpha \in \text{CS}} x^{\text{col}(\alpha)} y^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = \frac{qxy}{1 - q(x+y) - \frac{q^3xy}{1 - q^2(x+y) - \frac{q^5xy}{\dots}}}$$

We note that in [4, Corollary 4.6] the sequence of the coefficients of $(x+y)$ in the continued fraction (1.4) was inadvertently written q, q^3, q^5, \dots , where the correct sequence is q, q^2, q^3, \dots . We also note that there are similar results in [2].

The second main result of this paper is a finite version of Theorem 1.5 using q -Lommel polynomials. We first recall the connection between Bessel functions and Lommel polynomials.

The *Lommel polynomials* $R_{n,\nu}(z)$ are polynomials in z^{-1} defined by $R_{0,\nu}(z) = 1$, $R_{1,\nu}(z) = 2\nu/z$, and for $n \geq 0$,

$$(1.5) \quad R_{n+1,\nu}(z) = \frac{2(n+\nu)}{z} R_{n,\nu}(z) - R_{n-1,\nu}(z).$$

The Bessel functions and the Lommel polynomials are related by the recurrence

$$(1.6) \quad J_{\nu+n}(z) = R_{n,\nu}(z) J_{\nu}(z) - R_{n-1,\nu+1}(z) J_{\nu-1}(z).$$

Hurwitz's theorem [14, Theorem 6.5.4] says that the Bessel function $J_{\nu}(z)$ is obtained as a limit of the Lommel polynomials $R_{n,\nu+1}(z)$:

$$(1.7) \quad \lim_{n \rightarrow \infty} \frac{(z/2)^n R_{n,\nu+1}(z)}{\Gamma(n+\nu+1)} = (z/2)^{-\nu} J_{\nu}(z).$$

Thanks to Hurwitz's theorem one can regard Lommel polynomials as a finite analog of Bessel functions. Note that the ratio of Bessel functions in Kishore's theorem is the limit of a ratio of Lommel polynomials:

$$(1.8) \quad \frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \lim_{n \rightarrow \infty} \frac{R_{n,\nu+2}(z)}{R_{n+1,\nu+1}(z)}.$$

In this paper we study q -analogs of these ratios. Koelink and Swarttouw [21, (4.18)] introduced the following q -Lommel polynomials.

Definition 1.6. The q -Lommel polynomials $R_{m,\nu}(z; q)$ associated to the Jackson's third q -Bessel functions $J_{\nu}(z; q)$ are the Laurent polynomials in z defined by $R_{-1,\nu}(z; q) = 0$, $R_{0,\nu}(z; q) = 1$, and for $m \geq 0$,

$$(1.9) \quad R_{m+1,\nu}(z; q) = (z + z^{-1}(1 - q^{\nu+m})) R_{m,\nu}(z; q) - R_{m-1,\nu}(z; q).$$

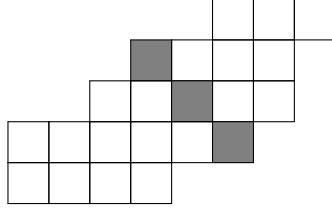
In this paper we will consider $J_{\nu}(z; q^{-1})$ and $R_{m,\nu}(z; q^{-1})$ instead of $J_{\nu}(z; q)$ and $R_{m,\nu}(z; q)$. Koelink and Swarttouw [21, (4.12), (4.24)] showed that these q -Lommel polynomials satisfy the following properties analogous to (1.6) and (1.7):

$$(1.10) \quad J_{\nu+m}(z; q^{-1}) = R_{m,\nu}(z; q^{-1}) J_{\nu}(z; q^{-1}) - R_{m-1,\nu+1}(z; q^{-1}) J_{\nu-1}(z; q^{-1}),$$

$$(1.11) \quad \lim_{m \rightarrow \infty} z^m R_{m,\nu}(z; q^{-1}) = \frac{(q^{-1}; q^{-1})_{\infty} z^{1-\nu}}{(z^2; q^{-1})_{\infty}} J_{\nu-1}(z; q^{-1}).$$

By (1.11) we have a q -analog of (1.8):

$$(1.12) \quad \lim_{n \rightarrow \infty} \frac{R_{n,\nu+2}(z; q^{-1})}{R_{n+1,\nu+1}(z; q^{-1})} = \frac{J_{\nu+1}(z; q^{-1})}{J_{\nu}(z; q^{-1})}.$$

FIGURE 2. A diagonal of a skew shape in $\text{CS}^{\leq 3}$.

Note that

$$(1.13) \quad \begin{aligned} \frac{J_{\nu+1}(z; q^{-1})}{J_{\nu}(z; q^{-1})} &= \frac{z}{1 - q^{-\nu-1}} \cdot \frac{{}_1\phi_1(0; q^{-\nu-2}; q^{-1}, q^{-1}z^2)}{{}_1\phi_1(0; q^{-\nu-1}; q^{-1}, q^{-1}z^2)} \\ &= \frac{-q^{\nu+1}z}{1 - q^{\nu+1}} \cdot \frac{{}_1\phi_1(0; q^{\nu+2}; q, q^{\nu+2}z^2)}{{}_1\phi_1(0; q^{\nu+1}; q, q^{\nu+1}z^2)}, \end{aligned}$$

where the last equality holds as formal power series in z whose coefficients are rational functions in q and q^{ν} . Therefore one may consider (1.12) as an equation in the ring of formal power series in z , q^{-1} , and $q^{-\nu}$, or in z , q , and q^{ν} .

Before stating our second main result we need one more definition.

Definition 1.7. For a connected skew shape α , a *diagonal* is the set of squares in row i and column j such that $i - j = k$ for a fixed (not necessary positive) integer k . Let $\text{CS}^{\leq m}$ denote the set of connected skew shapes in which every diagonal has at most m squares. See Figure 2.

Using Flajolet's theory of continued fractions [9] we show the following finite version of (1.4).

Proposition 1.8. For any positive integer m , we have

$$\sum_{\alpha \in \text{CS}^{\leq m}} x^{\text{col}(\alpha)} y^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = \frac{qxy}{1 - q(x+y) - \frac{q^3xy}{1 - q^2(x+y) - \dots - \frac{q^{2m+1}xy}{1 - q^{m+1}(x+y)}}}.$$

The second main result of this paper is the following finite version of Theorem 1.5.

Theorem 1.9. For any positive integer m , we have

$$\sum_{\alpha \in \text{CS}^{\leq m}} (q^{\nu}z^2)^{\text{col}(\alpha)} (q^{\nu})^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = -q^{\nu}z \frac{R_{m,\nu+2}(z; q^{-1})}{R_{m+1,\nu+1}(z; q^{-1})}.$$

If we take the limit $m \rightarrow \infty$ in Theorem 1.9, by (1.12), we obtain Theorem 1.5.

Note that Theorem 1.5 and (1.4) give

$$(1.14) \quad -q^{\nu}z \frac{J_{\nu+1}(z; q^{-1})}{J_{\nu}(z; q^{-1})} = \frac{q^{2\nu+1}z^2}{1 - q(q^{\nu} + q^{\nu}z^2) - \frac{q^{2\nu+3}z^2}{1 - q^2(q^{\nu} + q^{\nu}z^2) - \frac{q^{2\nu+5}z^2}{\dots}}}.$$

In Theorem 5.1 we show that this continued fraction is a generating function for moments of orthogonal polynomials of type R_I . We will show in Theorem 6.5 that the same ratio $J_{\nu+1}(z; q^{-1})/J_{\nu}(z; q^{-1})$ can also be expressed as a generating function for moments of usual orthogonal polynomials.

Fédou [8, Theorem 2] gave several combinatorial interpretations for the ratio, and inverse, of Jackson's third q -Bessel function. Barucci et al. [2] used steep convex polyominoes to give a combinatorial interpretation for a ratio of Jackson's first q -Bessel function. Barucci et al. [3, Theorem 4.3] used permutation statistics for another combinatorial interpretation for a ratio of Jackson's first q -Bessel function. They also gave interpretations for functions closely related to q -Bessel functions in both papers. Li [23, Proposition 4.5] gave a combinatorial meaning to the coefficients of the power series expansion of $1/J_0^{(1)}(z; q)$. Aval et al. [1] and Jin [16] considered the combinatorics of $1/J_0(z)$.

The remainder of this paper is organized as follows. In Section 2 we prove the q -Kishore theorem (Theorem 1.3) using Jackson's third q -Bessel function and its analogous result with the usual q base instead of q^{-1} . In Section 3 we prove another q -Kishore theorem using Jackson's first and second q -Bessel functions. In Section 4 we review basic results on orthogonal polynomials of type R_I and continued fractions. Using these results, in Section 5 we prove Theorem 1.9 and show that the ratio $J_{\nu+1}(z; q^{-1})/J_\nu(z; q^{-1})$ is a generating function for moments of orthogonal polynomials of type R_I . In Section 6 we show that this ratio can also be written as a generating function for moments of usual orthogonal polynomials. Finally in Section 7 we propose some open problems.

2. THE q -KISHORE THEOREM FOR JACKSON'S THIRD q -BESSEL FUNCTION

In this section we prove Theorem 1.3 and an analogous result, Theorem 2.6, with the usual q base instead of q^{-1} .

Kishore [20] proved Theorem 1.1 by finding a recurrence relation for the coefficient of z^n . He did this using differential equations for Bessel functions. We will use similar methods using q -differential equations to prove Theorem 1.3.

We now review some facts about q -derivatives. Recall that the q -derivative $D_q(f(x))$ is defined by

$$D_q(f(x)) = \frac{f(x) - f(qx)}{x - qx},$$

and it satisfies the following properties:

$$\begin{aligned} D_q(x^n) &= [n]_q x^{n-1}, \\ D_q(f(x)g(x)) &= D_q(f(x))g(x) + f(qx)D_q(g(x)), \\ D_q\left(\frac{1}{f(x)}\right) &= \frac{-D_q(f(x))}{f(x)f(qx)}. \end{aligned}$$

In order to prove Theorem 1.3 we introduce some notation. By (1.13), we have

$$\frac{J_{\nu+1}((1-q)z; q^{-1})}{J_\nu((1-q)z; q^{-1})} = \frac{-q^{\nu+1}z {}_1\phi_1(0; q^{\nu+2}; q, (1-q)^2 z^2 q^{\nu+2})}{[\nu+1]_q {}_1\phi_1(0; q^{\nu+1}; q, (1-q)^2 z^2 q^{\nu+1})}.$$

Let $x = qz^2$ and define $\theta_\nu(x)$, $F(x)$, and $\mu_n(q)$ by

$$\begin{aligned} \theta_\nu(x) &= {}_1\phi_1(0; q^{\nu+1}; q, (1-q)^2 x q^\nu) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(-1)^n (xq^\nu)^n (1-q)^{2n}}{(q; q)_n (q^{\nu+1}; q)_n}, \\ F(x) &= \frac{\theta_{\nu+1}(x)}{\theta_\nu(x)} = \sum_{n \geq 0} \mu_n(q) x^n, \end{aligned}$$

so that

$$(2.1) \quad \frac{J_{\nu+1}((1-q)z; q^{-1})}{J_{\nu}((1-q)z; q^{-1})} = -\frac{q^{\nu+1}z}{[\nu+1]_q} \sum_{n \geq 0} \mu_n(q) q^n z^{2n}.$$

We give a recurrence relation for $\mu_n(q)$.

Proposition 2.1. *We have*

$$\mu_0(q) = 1, \quad \mu_1(q) = \frac{q^{\nu}}{[\nu+1]_q[\nu+2]_q},$$

and for $n \geq 2$,

$$q^{-\nu}[\nu+1]_q[\nu+n+1]_q \mu_n(q) = \sum_{k=2}^{n-1} q^{\nu+k} \mu_{k-1}(q) \mu_{n-k}(q) + (1+q^{\nu+n}) \mu_{n-1}(q).$$

Proof. The values of μ_0 and μ_1 can be checked easily from the definition. We claim that for $n \geq 2$,

$$(2.2) \quad \frac{q^{-\nu}[\nu+1]_q}{x^{\nu+1}} D_q(x^{\nu+1}F(x)) = q^{\nu+1}F(x)F(qx) + (1-q^{\nu+1})F(x) + [\nu+1]_q^2/xq^{\nu},$$

from which the proposition follows by equating the coefficients of x^{n-1} on both sides.

Note that

$$\begin{aligned} D_q(x^{\nu+1}\theta_{\nu+1}(x)) &= x^{\nu} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(-1)^n (xq^{\nu+1})^n (1-q)^{2n}}{(q; q)_n (q^{\nu+2}; q)_{n-1}} \\ &= x^{\nu} [\nu+1]_q \theta_{\nu}(qx) \end{aligned}$$

and

$$\begin{aligned} D_q(\theta_{\nu}(x)) &= \frac{\theta_{\nu}(x) - \theta_{\nu}(qx)}{x - qx} = \frac{1}{(1-q)x} \sum_{n=1}^{\infty} q^{\binom{n}{2}} \frac{(-1)^n (xq^{\nu})^n (1-q)^{2n}}{(q; q)_{n-1} (q^{\nu+1}; q)_n} \\ &= \frac{-(1-q)q^{\nu}}{1-q^{\nu+1}} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(-1)^n (xq^{\nu+1})^n (1-q)^{2n}}{(q; q)_n (q^{\nu+2}; q)_n} \\ &= \frac{-q^{\nu}}{[\nu+1]_q} \theta_{\nu+1}(x). \end{aligned}$$

By the properties of q -derivative and the above two identities, we have

$$\begin{aligned} D_q(x^{\nu+1}F(x)) &= D_q\left(\frac{x^{\nu+1}\theta_{\nu+1}(x)}{\theta_{\nu}(x)}\right) \\ &= D_q\left(\frac{1}{\theta_{\nu}(x)}\right) x^{\nu+1}\theta_{\nu+1}(x) + \frac{1}{\theta_{\nu}(qx)} D_q(x^{\nu+1}\theta_{\nu+1}(x)) \\ &= \frac{1}{\theta_{\nu}(qx)\theta_{\nu}(x)} \frac{\theta_{\nu}(qx) - \theta_{\nu}(x)}{x - qx} x^{\nu+1}\theta_{\nu+1}(x) + \frac{x^{\nu}[\nu+1]_q \theta_{\nu}(qx)}{\theta_{\nu}(qx)} \\ &= \frac{1}{\theta_{\nu}(qx)\theta_{\nu}(x)} \frac{q^{\nu}}{[\nu+1]_q} \theta_{\nu+1}(x) x^{\nu+1}\theta_{\nu+1}(x) + x^{\nu}[\nu+1]_q \\ &= \frac{1}{\theta_{\nu}(qx)\theta_{\nu}(x)} \frac{q^{\nu}}{[\nu+1]_q} \theta_{\nu+1}(x) x^{\nu+1} (q^{\nu+1}\theta_{\nu+1}(qx) + (1-q^{\nu+1})\theta_{\nu}(qx)) + x^{\nu}[\nu+1]_q \\ &= \frac{q^{2\nu+1}}{[\nu+1]_q} x^{\nu+1}F(x)F(qx) + q^{\nu}(1-q)x^{\nu+1}F(x) + x^{\nu}[\nu+1]_q. \end{aligned}$$

Then (2.2) follows upon multiplying by $q^{-\nu}[\nu+1]_q/x^{\nu+1}$. \square

We are now ready to prove Theorem 1.3, which is stated again below.

Theorem 2.2. *For any ν (not a negative integer) we have*

$$\frac{J_{\nu+1}((1-q)z; q^{-1})}{J_{\nu}((1-q)z; q^{-1})} = - \sum_{n=1}^{\infty} \frac{N_{n,\nu}(q)}{D_{n,\nu}(q)} q^{(\nu+1)n} z^{2n-1},$$

where each $N_{n,\nu}(q)$ is a polynomial in q, q^{ν} and $[\nu]_q$ with nonnegative integer coefficients and

$$D_{n,\nu}(q) = \prod_{k=1}^n [k + \nu]_q^{\lfloor n/k \rfloor}.$$

Proof. By (2.1), Theorem 1.3 can be rewritten as

$$\frac{q^{\nu+1}z}{[\nu+1]_q} \sum_{n \geq 0} \mu_n(q) q^n z^{2n} = \sum_{n=1}^{\infty} \frac{N_{n,\nu}(q)}{D_{n,\nu}(q)} q^{(\nu+1)n} z^{2n-1} = \sum_{n=0}^{\infty} \frac{N_{n+1,\nu}(q)}{D_{n+1,\nu}(q)} q^{(\nu+1)(n+1)} z^{2n+1},$$

or equivalently, for each $n \geq 0$,

$$(2.3) \quad \frac{\mu_n(q)}{q^{n\nu}[\nu+1]_q} = \frac{N_{n+1,\nu}(q)}{D_{n+1,\nu}(q)}.$$

Let $d_n = D_{n+1,\nu}(q) = \prod_{k=1}^{n+1} [\nu+k]_q^{\lfloor (n+1)/k \rfloor}$ and $\beta_n = N_{n+1,\nu}(q) = \mu_n(q) d_n / q^{n\nu} [\nu+1]_q$. Then we must show that β_n is a polynomial in q, q^{ν} , and $[\nu]_q$ with nonnegative integer coefficients.

We prove this by induction on n . If $n=0$, we have $\beta_0=1$. If $n=1$, we also have $\beta_1=1$ since $\mu_1(q) = q^{\nu}/[\nu+1]_q[\nu+2]_q$ and $d_1 = [\nu+1]_q^2[\nu+2]_q$. Let $n \geq 2$ and suppose that the claim is true for all $1 \leq k < n$. By multiplying both sides of the equation in Proposition 2.1 by $d_n/q^{(n-1)\nu}[\nu+1]_q^2[\nu+n+1]_q$ we obtain

$$\beta_n = \sum_{k=2}^{n-1} q^{\nu+k} \beta_{k-1} \beta_{n-k} \frac{d_n}{d_{k-1} d_{n-k} [\nu+n+1]_q} + (1+q^{\nu+n}) \beta_{n-1} \frac{d_n}{d_{n-1} [\nu+n+1]_q}.$$

It is straightforward to check that for $2 \leq k \leq n-1$,

$$\frac{d_n}{d_{k-1} d_{n-k} [\nu+n+1]_q}, \quad \frac{d_n}{d_{n-1} [\nu+n+1]_q}$$

are polynomials in q, q^{ν} , and $[\nu]_q$ with nonnegative integer coefficients using the fact $[\nu+j]_q = [\nu]_q + q^{\nu}[j]_q$. Therefore β_n is also a polynomial with nonnegative coefficients, which completes the proof by induction. \square

Remark 2.3. Delest and Fédou [6] in fact considered the following modified q -Bessel functions. For a nonnegative integer ν , let

$$T_{\nu}(x; q) = \sum_{n \geq 0} \frac{(-1)^n q^{\binom{n+\nu}{2}} x^{n+\nu}}{[n]_q! [n+\nu]_q!} = \frac{q^{\binom{\nu}{2}} ((1-q)x)^{\nu}}{(q; q)_{\nu}} {}_1\phi_1(0; q^{\nu+1}; q, (1-q)^2 q^{\nu} x),$$

where $[n]_q! = [1]_q [2]_q \cdots [n]_q$. Then the original form of the conjecture in [6, Conjecture 9] states that

$$\frac{T_{\nu+1}(x; q)}{T_{\nu}(x; q)} = \sum_{n=1}^{\infty} \frac{N_{n,\nu}(q)}{D_{n,\nu}(q)} x^n,$$

where $N_{n,\nu}(q)$ and $D_{n,\nu}(q)$ are as in Theorem 1.3. It is straightforward to check that

$$\frac{T_{\nu+1}(x; q)}{T_{\nu}(x; q)} = -x^{1/2}q^{-1/2} \frac{J_{\nu+1}((1-q)x^{1/2}q^{-1/2}; q^{-1})}{J_{\nu}((1-q)x^{1/2}q^{-1/2}; q^{-1})},$$

which shows that their conjecture follows from Theorem 1.3.

For the rest of this section as corollaries of Theorem 1.3 we give two more q -analogs of Kishore's theorem using the usual q base instead of q^{-1} . Replacing q by q^{-1} in (2.1), we obtain

$$(2.4) \quad \frac{J_{\nu+1}((1-q^{-1})z; q)}{J_{\nu}((1-q^{-1})z; q)} = -\frac{q^{-\nu-1}z}{[\nu+1]_{q^{-1}}} \sum_{n \geq 0} \mu_n(q^{-1})q^{-n}z^{2n}.$$

If we replace z by $-qz$ in (2.4) we obtain

$$(2.5) \quad \frac{J_{\nu+1}((1-q)z; q)}{J_{\nu}((1-q)z; q)} = \frac{1}{[\nu+1]_q} \sum_{n \geq 0} \mu_n(q^{-1})q^n z^{2n+1}.$$

Therefore by replacing q by q^{-1} in Proposition 2.1 and using the fact $[\nu+1]_{q^{-1}} = q^{-\nu}[\nu+1]_q$ we obtain the following recurrence relation for $\mu_n(q^{-1})$.

Proposition 2.4. *We have*

$$\mu_0(q^{-1}) = 1, \quad \mu_1(q^{-1}) = \frac{q^{\nu+1}}{[\nu+1]_q[\nu+2]_q},$$

and for $n \geq 2$,

$$[\nu+1]_q[\nu+n+1]_q \mu_n(q^{-1}) = \sum_{k=2}^{n-1} q^{n-k} \mu_{k-1}(q^{-1}) \mu_{n-k}(q^{-1}) + (1+q^{\nu+n}) \mu_{n-1}(q^{-1}).$$

Using the recurrences in Propositions 2.1 and 2.4 the following relation between $\mu_n(q^{-1})$ and $\mu_n(q)$ is easily shown by induction.

Proposition 2.5. *For $n \geq 1$,*

$$\mu_n(q^{-1}) = q^{1-(n-1)\nu} \mu_n(q).$$

Now we can prove a second q -analog of Kishore's theorem.

Theorem 2.6. *For any ν (not a negative integer) we have*

$$\frac{J_{\nu+1}((1-q)z; q)}{J_{\nu}((1-q)z; q)} = (1-q)z + \sum_{n \geq 1} \frac{N_{n,\nu}(q)}{D_{n,\nu}(q)} q^{n+\nu} z^{2n-1},$$

where $N_{n,\nu}(q)$ and $D_{n,\nu}(q)$ are the same polynomials given in Theorem 1.3.

Proof. By Proposition 2.5 and (2.3), we have

$$\begin{aligned}
\frac{J_{\nu+1}((1-q)z; q)}{J_{\nu}((1-q)z; q)} &= \frac{1}{[\nu+1]_q} \left(z + \sum_{n \geq 1} \mu_n(q^{-1}) q^n z^{2n+1} \right) \\
&= \frac{1}{[\nu+1]_q} \left(z + \sum_{n \geq 1} q^{n+1-(n-1)\nu} \mu_n(q) z^{2n+1} \right) \\
&= \frac{1}{[\nu+1]_q} \left(z - q^{1+\nu} z + \sum_{n \geq 0} q^{n+1-(n-1)\nu} \mu_n(q) z^{2n+1} \right) \\
&= (1-q)z + \sum_{n \geq 0} \frac{N_{n+1, \nu}(q)}{D_{n+1, \nu}(q)} q^{n+1+\nu} z^{2n+1} \\
&= (1-q)z + \sum_{n \geq 1} \frac{N_{n, \nu}(q)}{D_{n, \nu}(q)} q^{n+\nu} z^{2n-1},
\end{aligned}$$

as desired. \square

Comparing Theorems 1.3 and 2.6 we obtain the following corollary.

Corollary 2.7. *For any ν (not a negative integer) we have*

$$\frac{J_{\nu+1}(-(1-q)q^{-\nu/2}z; q^{-1})}{J_{\nu}(-(1-q)q^{-\nu/2}z; q^{-1})} = q^{-\nu/2} \frac{J_{\nu+1}((1-q)z; q)}{J_{\nu}((1-q)z; q)} - (1-q)q^{-\nu/2}z.$$

Proof. By Theorems 1.3 and 2.6 we have

$$\begin{aligned}
\frac{J_{\nu+1}(-(1-q)q^{-\nu/2}z; q^{-1})}{J_{\nu}(-(1-q)q^{-\nu/2}z; q^{-1})} &= \sum_{n \geq 1} \frac{N_{n, \nu}(q)}{D_{n, \nu}(q)} q^{(\nu+1)n} q^{-\frac{\nu}{2}(2n-1)} z^{2n-1} \\
&= q^{-\nu/2} \sum_{n \geq 1} \frac{N_{n, \nu}(q)}{D_{n, \nu}(q)} q^{\nu+n} z^{2n-1} \\
&= q^{-\nu/2} \left(\frac{J_{\nu+1}((1-q)z; q)}{J_{\nu}((1-q)z; q)} - (1-q)z \right),
\end{aligned}$$

as desired. \square

We note that Corollary 2.7 can also be proved directly by finding the coefficients of z^{2k+1} on both sides.

Combining Definition 1.2 and (1.13) gives

$$(2.6) \quad \frac{J_{\nu+1}((1-q)z; q^{-1})}{J_{\nu}((1-q)z; q^{-1})} = -q^{1/2} \frac{J_{\nu+1}(q^{(\nu+1)/2}(1-q)z; q)}{J_{\nu}(q^{\nu/2}(1-q)z; q)}.$$

By Theorem 1.3 and (2.6) we obtain a third q -analog of Kishore's theorem.

Theorem 2.8. *For any ν (not a negative integer) we have*

$$\frac{J_{\nu+1}((1-q)q^{1/2}z; q)}{J_{\nu}((1-q)z; q)} = q^{(\nu-1)/2} \sum_{n \geq 1} \frac{N_{n, \nu}(q)}{D_{n, \nu}(q)} q^n z^{2n-1},$$

where $N_{n, \nu}(q)$ and $D_{n, \nu}(q)$ are the same polynomials given in Theorem 1.3.

3. THE q -KISHORE THEOREM FOR JACKSON'S FIRST q -BESSEL FUNCTION

In this section we give a q -analog of Kishore's theorem, which uses Jackson's first q -Bessel function. There is an identical theorem for Jackson's second q -Bessel function. The arguments are similar to those in the previous section.

Definition 3.1. Jackson's first q -Bessel function $J_\nu^{(1)}(z; q)$ and second q -Bessel function $J_\nu^{(2)}(z; q)$ are defined by

$$(3.1) \quad J_\nu^{(1)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} (z/2)^\nu {}_2\phi_1(0, 0; q^{\nu+1}; q, -z^2/4),$$

$$(3.2) \quad J_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} (z/2)^\nu {}_0\phi_1(-; q^{\nu+1}; q, -q^{\nu+1}z^2/4).$$

From the definitions it follows easily that, for $k = 1, 2$,

$$\lim_{q \rightarrow 1^-} J_\nu^{(k)}(x(1-q); q) = J_\nu(x).$$

There is a simple connection between $J_\nu^{(1)}(z; q)$ and $J_\nu^{(2)}(z; q)$, see [14, Theorem 14.1.3]:

$$(3.3) \quad (-z^2/4; q)_\infty J_\nu^{(1)}(z; q) = J_\nu^{(2)}(z; q).$$

By (3.3), in order to compute the ratio $J_{\nu+1}^{(k)}(z; q)/J_\nu^{(k)}(z; q)$ for $k = 1, 2$, it suffices to consider only the first q -Bessel function. This is the reason only $J_\nu^{(1)}(z; q)$ appears in this section.

Theorem 3.2. For any ν (not a negative integer) we have

$$(3.4) \quad \frac{J_{\nu+1}^{(1)}((1-q)z; q)}{J_\nu^{(1)}((1-q)z; q)} = \sum_{n=1}^{\infty} \frac{N_{n,\nu}^{(1)}(q)}{D_{n,\nu}(q)} q^{(\nu+1)(n-1)} \left(\frac{z}{2}\right)^{2n-1},$$

where each $N_{n,\nu}^{(1)}(q)$ is a polynomial in q, q^ν and $[\nu]_q$ with nonnegative integer coefficients and

$$D_{n,\nu}(q) = \prod_{k=1}^n [k + \nu]_q^{\lfloor n/k \rfloor}.$$

As in the previous section we introduce some notation. By definition we have

$$\frac{J_{\nu+1}^{(1)}((1-q)z; q)}{J_\nu^{(1)}((1-q)z; q)} = \frac{z/2} {[\nu+1]_q} \frac{{}_2\phi_1(0, 0; q^{\nu+2}; q, -(1-q)^2z^2/4)}{{}_2\phi_1(0, 0; q^{\nu+1}; q, -(1-q)^2z^2/4)}.$$

Let $x = z^2/4$ and define $\theta_\nu^{(1)}(x)$, $F^{(1)}(x)$, and $\mu_n^{(1)}(q)$ by

$$\begin{aligned} \theta_\nu^{(1)}(x) &= {}_2\phi_1(0, 0; q^{\nu+1}; q, -(1-q)^2x), \\ F^{(1)}(x) &= \frac{\theta_{\nu+1}^{(1)}(x)}{\theta_\nu^{(1)}(x)} = \sum_{n=0}^{\infty} \mu_n^{(1)}(q)x^n, \end{aligned}$$

so that

$$(3.5) \quad \frac{J_{\nu+1}^{(1)}((1-q)z; q)}{J_\nu^{(1)}((1-q)z; q)} = \frac{z/2} {[\nu+1]_q} \sum_{n \geq 0} \mu_n^{(1)}(q)z^{2n}.$$

We first show a recurrence relation for μ_n .

Proposition 3.3. *We have $\mu_0^{(1)}(q) = 1$ and for $n \geq 1$,*

$$q^{-\nu-1}[\nu+1]_q[\nu+n+1]_q\mu_n^{(1)}(q) = \sum_{k=0}^{n-1} q^k \mu_k^{(1)}(q)\mu_{n-k-1}^{(1)}(q).$$

Proof. We claim that

$$\frac{[\nu+1]_q}{(qx)^{\nu+1}} D_q(x^{\nu+1}F(x)) = F(x)F(qx) + \frac{[\nu+1]^2}{q^{\nu+1}x},$$

from which the proposition follows by equating the coefficients of x^{n-1} on both sides.

The q -product rule implies

$$\begin{aligned} \frac{[\nu+1]_q}{(qx)^{\nu+1}} D_q(x^{\nu+1}F(x)) &= \frac{[\nu+1]_q}{(qx)^{\nu+1}} D_q\left(\frac{x^{\nu+1}\theta_{\nu+1}(x)}{\theta_\nu(x)}\right) \\ (3.6) \qquad \qquad \qquad &= \frac{[\nu+1]_q}{(qx)^{\nu+1}} \frac{D_q(x^{\nu+1}\theta_{\nu+1}(x))}{\theta_\nu(x)} + [\nu+1]_q\theta_{\nu+1}(qx)D_q\left(\frac{1}{\theta_\nu(x)}\right). \end{aligned}$$

Since

$$\frac{D_q(x^{\nu+1}\theta_{\nu+1}(x))}{x^{\nu+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n(1-q)^{2n}x^{n-1}}{(q;q)_n(q^{\nu+2};q)_n} \frac{1-q^{\nu+n+1}}{1-q} = [\nu+1]_q\theta_\nu(x)/x$$

and

$$\begin{aligned} D_q(1/\theta_\nu(x)) &= (1/\theta_\nu(x) - 1/\theta_\nu(qx))/(1-q)x \\ &= \frac{\theta_\nu(qx) - \theta_\nu(x)}{(1-q)x\theta_\nu(x)\theta_\nu(qx)} \\ &= -\frac{1}{\theta_\nu(x)\theta_\nu(qx)} \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1} (1-q)^{2n} (1-q^n)}{(q;q)_n (q^{\nu+1};q)_n (1-q)} \\ &= \frac{1}{\theta_\nu(x)\theta_\nu(qx)} \frac{1}{[\nu+1]_q} \sum_{n=0}^{\infty} \frac{x^n (1-q)^{2n}}{(q;q)_n (q^{\nu+2};q)_n} \\ &= \frac{1}{[\nu+1]_q} \frac{\theta_{\nu+1}(x)}{\theta_\nu(x)\theta_\nu(xq)} \end{aligned}$$

we see that (3.6) becomes

$$\begin{aligned} \frac{[\nu+1]_q}{(qx)^{\nu+1}} D_q(x^{\nu+1}F(x)) &= \frac{[\nu+1]_q^2}{q^{\nu+1}x} + \frac{\theta_{\nu+1}(x)\theta_{\nu+1}(qx)}{\theta_\nu(x)\theta_\nu(xq)} \\ &= \frac{[\nu+1]_q^2}{q^{\nu+1}x} + F(x)F(qx) \end{aligned}$$

as required. \square

Proof of Theorem 3.2. By (3.5) we have

$$\frac{1}{[\nu+1]_q} \sum_{n \geq 0} \mu_n^{(1)}(q) \left(\frac{z}{2}\right)^{2n} = \sum_{n=1}^{\infty} \frac{N_{n,\nu}^{(1)}(q)}{D_{n,\nu}^{(1)}(q)} q^{(\nu+1)(n-1)} \left(\frac{z}{2}\right)^{2n-2}.$$

Let $d_n = D_{n+1,\nu}(q) = \prod_{k=1}^{n+1} [\nu+k]_q^{\lfloor (n+1)/k \rfloor}$ and $\beta_n = N_{n+1,\nu}^{(1)}(q)$. Then we need to show that

$$\beta_n = \frac{\mu_n^{(1)}(q)d_n}{q^{(\nu+1)n}[\nu+1]_q}$$

is a polynomial in q, q^ν and $[\nu]_q$ with nonnegative integer coefficients. We prove this by induction on n . It is true for $n = 0$ since $\beta_0 = 1$.

For the inductive step let $n \geq 1$ and suppose that β_k is a polynomial in q, q^ν , and $[\nu]_q$ for all $0 \leq k < n$. Multiplying both sides of the equation in Proposition 3.3 by $d_n/q^{(\nu+1)(n-1)}[\nu+1]_q^2$, we obtain

$$(3.7) \quad \beta_n = \sum_{k=0}^{n-1} \frac{q^k d_n}{d_k d_{n-1-k} [\nu+n+1]_q} \beta_k \beta_{n-1-k}.$$

It is easy to check that for $0 \leq k \leq n-1$,

$$\frac{d_n}{d_k d_{n-1-k} [\nu+n+1]_q}$$

is a polynomial in q, q^ν and $[\nu]_q$ with nonnegative integer coefficients. Then by induction hypothesis (3.7) shows that β_n is also such a polynomial, completing the proof. \square

As in the previous section we can also obtain a similar result using the base q^{-1} . Replacing q by q^{-1} in (3.5), we obtain

$$(3.8) \quad \frac{J_{\nu+1}^{(1)}((1-q^{-1})z; q^{-1})}{J_\nu^{(1)}((1-q^{-1})z; q^{-1})} = \frac{q^\nu z/2}{[\nu+1]_q} \sum_{n \geq 0} \mu_n^{(1)}(q^{-1}) z^{2n}.$$

Using Proposition 3.3 it is easy to show that for $n \geq 0$,

$$(3.9) \quad \mu_n^{(1)}(q) = q^n \mu_n^{(1)}(q^{-1}).$$

Replacing z by $-qz$ in (3.8) and using (3.9) gives

$$(3.10) \quad \frac{J_{\nu+1}^{(1)}((1-q)z; q^{-1})}{J_\nu^{(1)}((1-q)z; q^{-1})} = -\frac{q^{\nu+1} z/2}{[\nu+1]_q} \sum_{n \geq 0} \mu_n^{(1)}(q) q^n z^{2n}.$$

By (3.5) and (3.10),

$$(3.11) \quad \frac{J_{\nu+1}^{(1)}((1-q)z; q^{-1})}{J_\nu^{(1)}((1-q)z; q^{-1})} = -q^{\nu+\frac{1}{2}} \frac{J_{\nu+1}^{(1)}((1-q)q^{1/2}z; q)}{J_\nu^{(1)}((1-q)q^{1/2}z; q)}.$$

Therefore by Theorem 3.2 and (3.11) we obtain an analogous result of Theorem 3.2.

Theorem 3.4. *For any ν (not a negative integer) we have*

$$(3.12) \quad \frac{J_{\nu+1}^{(1)}((1-q)z; q^{-1})}{J_\nu^{(1)}((1-q)z; q^{-1})} = -\sum_{n=1}^{\infty} \frac{N_{n,\nu}^{(1)}(q)}{D_{n,\nu}(q)} q^{\nu n-1} \left(\frac{z}{2}\right)^{2n-1},$$

where $N_{n,\nu}^{(1)}(q)$ and $D_{n,\nu}(q)$ are the same polynomials given in Theorem 3.2.

4. ORTHOGONAL POLYNOMIALS OF TYPE R_I AND CONTINUED FRACTIONS

In this section we review basic results in [15] and [19] on orthogonal polynomials of type R_I and continued fractions, which will be used in the next section. We begin by introducing some notation for continued fractions.

Definition 4.1. For sequences a_i and b_i , let

$$\mathbf{K}_{i=0}^m \left(\frac{a_i}{b_i} \right) = \frac{a_0}{b_0 + \frac{a_1}{b_1 + \dots + \frac{a_m}{b_m}}}, \quad \mathbf{K}_{i=0}^{\infty} \left(\frac{a_i}{b_i} \right) = \frac{a_0}{b_0 + \frac{a_1}{b_1 + \dots}}.$$

The following lemma will be used later.

Lemma 4.2. For any sequences $\{a_i : 0 \leq i \leq m\}$, $\{b_i : 0 \leq i \leq m\}$, and $\{c_i : -1 \leq i \leq m\}$, we have

$$\mathbf{K}_{i=0}^m \left(\frac{a_i}{b_i} \right) = \frac{1}{c_{-1}} \mathbf{K}_{i=0}^m \left(\frac{a_i c_{i-1} c_i}{b_i c_i} \right).$$

Proof. By multiplying c_i to by the numerator and denominator of the i th fraction, we obtain

$$\frac{a_0}{b_0 + \frac{a_1}{b_1 + \dots + \frac{a_m}{b_m}}} = \frac{a_0 c_0}{b_0 c_0 + \frac{a_1 c_0 c_1}{b_1 c_1 + \dots + \frac{a_m c_{m-1} c_m}{b_m c_m}}},$$

which is equivalent to the equation in the lemma. \square

Ismail and Masson [15] introduced orthogonal polynomials of type R_I generalizing the usual orthogonal polynomials.

Definition 4.3. A family of polynomials $p_n(x)$, $n \geq 0$, is called (*monic*) *orthogonal polynomials of type R_I* if they satisfy the three term recurrence relation: $p_{-1}(x) = 0$, $p_0(x) = 1$, and for $n \geq 0$,

$$p_{n+1}(x) = (x - b_n)p_n(x) - (a_n x + \lambda_n)p_{n-1}(x),$$

for some sequences $b = \{b_k\}_{k \geq 0}$, $a = \{a_k\}_{k \geq 0}$, and $\lambda = \{\lambda_k\}_{k \geq 0}$. In this case we say that $p_n(x)$ are the orthogonal polynomials of type R_I determined by the sequences b, a, λ .

The usual (monic) orthogonal polynomials are orthogonal polynomials of type R_I determined by sequences b, a, λ with $a_n = 0$ for all n . If $\lambda_n = 0$ for all n , then the orthogonal polynomials of type R_I becomes orthogonal Laurent polynomials, see [17, 18] for more details on orthogonal Laurent polynomials. In the next section we will see that the q -Lommel polynomials $\{R_{n,\nu}(x)\}_{n \geq 0}$ are orthogonal polynomials type R_I with $\lambda_n = 0$.

For any sequence $a = \{a_k\}_{k \geq 0}$, let $\delta a = \{\delta a_k\}_{k \geq 0}$ denote the sequence obtained by shifting the index up by 1, i.e., $\delta a_k = a_{k+1}$. Given orthogonal polynomials $p_n(x)$ of type R_I we consider two other sequences $\{\delta p_n(x)\}$ and $\{p_n^*(x)\}$ of polynomials associated to it as follows.

Definition 4.4. Let $\{p_n(x)\}$ be the orthogonal polynomials of type R_I determined by the sequences b, a, λ . We define $\{\delta p_n(x)\}_{n \geq 0}$ to be the orthogonal polynomials of type R_I determined by $\delta b, \delta a, \delta \lambda$, and define

$$p_n^*(x) = x^n p_n(1/x).$$

We will see that the quantities $\mu_n^{\leq m}(b, a, \lambda)$ and $\mu_n(b, a, \lambda)$ in the following definition are closely related to orthogonal polynomials of type R_I .

Definition 4.5. Let $b = \{b_k\}_{k \geq 0}$, $a = \{a_k\}_{k \geq 0}$, and $\lambda = \{\lambda_k\}_{k \geq 0}$ be sequences. For $m \geq 0$, we define $\mu_n^{\leq m}(b, a, \lambda)$ and $\mu_n(b, a, \lambda)$ by

$$\begin{aligned} \sum_{n \geq 0} \mu_n^{\leq m}(b, a, \lambda) z^n &= \frac{1}{1 - b_0 z - \frac{a_1 z + \lambda_1 z^2}{1 - b_1 z - \frac{a_2 z + \lambda_2 z^2}{1 - b_2 z - \dots - \frac{a_m z + \lambda_m z^2}{1 - b_m z}}}} \\ &= \frac{1}{-a_0 z - \lambda_0 z^2} \mathbf{K}_{i=0}^m \left(\frac{-a_i z - \lambda_i z^2}{1 - b_i z} \right), \\ \sum_{n \geq 0} \mu_n(b, a, \lambda) z^n &= \frac{1}{1 - b_0 z - \frac{a_1 z + \lambda_1 z^2}{1 - b_1 z - \frac{a_2 z + \lambda_2 z^2}{1 - b_2 z - \dots}}} \\ &= \frac{1}{-a_0 z - \lambda_0 z^2} \mathbf{K}_{i=0}^{\infty} \left(\frac{-a_i z - \lambda_i z^2}{1 - b_i z} \right). \end{aligned}$$

The continued fractions in the above definition are formal power series in z whose coefficients are polynomials in the elements of b , a , and λ . The infinite continued fraction always converges in this formal power series ring.

The following theorem can be proved using standard techniques in continued fractions, see [19] for a proof. The quantities $\mu_n(b, a, \lambda)$ in this theorem are called the *moments* of the orthogonal polynomials of type R_I . This result for the usual orthogonal polynomials, the case $a_n = 0$, is well known and Viennot [28] developed Flajolet's [9] combinatorial theory in this case using Motzkin paths. Kamioka [18, Proof of Lemma 3.3] gave a combinatorial model for the case $\lambda_n = 0$ using Schröder paths. Flajolet's theory can also be applied to the general case. A combinatorial model for the general case is given in [19] using certain lattice paths generalizing both Motzkin paths and Schröder paths.

Proposition 4.6. *Let $p_n(x)$ be the type R_I orthogonal polynomials determined by the sequences b , a , and λ . Then for $m \geq 0$,*

$$\begin{aligned} \frac{\delta p_m^*(x)}{p_{m+1}^*(x)} &= \sum_{n \geq 0} \mu_n^{\leq m}(b, a, \lambda) x^n, \\ \lim_{m \rightarrow \infty} \frac{\delta p_m^*(x)}{p_{m+1}^*(x)} &= \sum_{n \geq 0} \mu_n(b, a, \lambda) x^n. \end{aligned}$$

5. RATIOS OF q -LOMMEL POLYNOMIALS

In this section we prove Theorem 1.9, which gives a combinatorial interpretation for the ratio $R_{m, \nu+2}(z; q^{-1})/R_{m+1, \nu+1}(z; q^{-1})$ of q -Lommel polynomials. We also show that the ratio $J_{\nu+1}(z; q^{-1})/J_{\nu}(z; q^{-1})$ is a generating function for moments of orthogonal polynomials of type R_I .

We first show that the q -Lommel polynomials $R_{m, \nu}(x; q)$ give rise to orthogonal polynomials of type R_I . Then using Proposition 4.6 we interpret the ratio of q -Lommel polynomials as a continued fraction. We show that the continued fraction is a generating function for Motzkin paths of bounded height. Finally, we use a bijection between Motzkin paths and connection skew shapes to obtain the theorem.

Define *modified q -Lommel polynomials* $\tilde{R}_{m,\nu}(x; q)$ by

$$\tilde{R}_{m,\nu}(x; q) = \frac{x^{m/2}}{(q^\nu; q)_m} R_{m,\nu}(x^{-1/2}; q).$$

Then, by (1.9), we have $\tilde{R}_{0,\nu}(x; q) = 1$, $\tilde{R}_{-1,\nu}(x; q) = 0$, and for $m \geq 0$,

$$\tilde{R}_{m+1,\nu}(x; q) = \left(x + \frac{1}{1 - q^{\nu+m}} \right) \tilde{R}_{m,\nu}(x; q) - \frac{x}{(1 - q^{\nu+m-1})(1 - q^{\nu+m})} \tilde{R}_{m-1,\nu}(x; q),$$

which is equivalent to the recurrence in [21, (4.20)]. By replacing q by q^{-1} in the above recurrence we obtain

$$(5.1) \quad \tilde{R}_{m+1,\nu}(x; q^{-1}) = \left(x - \frac{q^{\nu+m}}{1 - q^{\nu+m}} \right) \tilde{R}_{m,\nu}(x; q^{-1}) - \frac{q^{2\nu+2m-1}x}{(1 - q^{\nu+m-1})(1 - q^{\nu+m})} \tilde{R}_{m-1,\nu}(x; q^{-1}).$$

Theorem 5.1. *Let $b = \{b_n\}_{n \geq 0}$, $a = \{a_n\}_{n \geq 0}$, and $\lambda = \{\lambda_n\}_{n \geq 0}$, where*

$$b_n = \frac{q^{\nu+n+1}}{1 - q^{\nu+n+1}}, \quad a_n = \frac{q^{2\nu+2n+1}}{(1 - q^{\nu+n})(1 - q^{\nu+n+1})}, \quad \lambda_n = 0.$$

Then

$$(5.2) \quad \frac{R_{m,\nu+2}(z; q^{-1})}{R_{m+1,\nu+1}(z; q^{-1})} = \frac{q^{\nu+1}z}{q^{\nu+1} - 1} \sum_{n \geq 0} \mu_n^{\leq m}(b, a, \lambda) z^{2n},$$

$$(5.3) \quad \frac{J_{\nu+1}(z; q^{-1})}{J_\nu(z; q^{-1})} = \frac{q^{\nu+1}z}{q^{\nu+1} - 1} \sum_{n \geq 0} \mu_n(b, a, \lambda) z^{2n}.$$

Proof. Let $p_n(x) = \tilde{R}_{n,\nu+1}(x; q^{-1})$. Then by (5.1), $p_n(x)$ are orthogonal polynomials of type R_I determined by b, a , and λ , and we have

$$p_m^*(x) = x^m p_m(1/x) = x^m \tilde{R}_{m,\nu+1}(1/x; q^{-1}) = \frac{x^{m/2}}{(q^{-\nu-1}; q^{-1})_m} R_{m,\nu+1}(x^{1/2}; q^{-1}),$$

$$\delta p_m^*(x) = \frac{x^{m/2}}{(q^{-\nu-2}; q^{-1})_m} R_{m,\nu+2}(x^{1/2}; q^{-1}).$$

By Proposition 4.6,

$$\sum_{n \geq 0} \mu_n^{\leq m}(b, a, \lambda) x^n = \frac{\delta p_m^*(x)}{p_{m+1}^*(x)} = (1 - q^{-\nu-1}) x^{-1/2} \frac{R_{m,\nu+2}(x^{1/2}; q^{-1})}{R_{m+1,\nu+1}(x^{1/2}; q^{-1}),$$

which gives the first identity with $z = x^{1/2}$. By taking the limit and using (1.12) we obtain the second identity. \square

Now we review Flajolet's theory [9] on continued fraction expressions for Motzkin path generating functions.

Definition 5.2. A *Motzkin path* is a lattice path from $(0, 0)$ to $(n, 0)$ consisting of up steps $(1, 1)$, down steps $(1, -1)$, and horizontal steps $(1, 0)$ that never goes below the x -axis. A *2-Motzkin path* is a Motzkin path in which every horizontal step is colored red or blue. The *height* of a 2-Motzkin path is the largest integer y for which (x, y) is a point in the path.

Denote by Motz_2 the set of all 2-Motzkin paths and by $\text{Motz}_2^{\leq m}$ the set of all 2-Motzkin paths with height at most m .

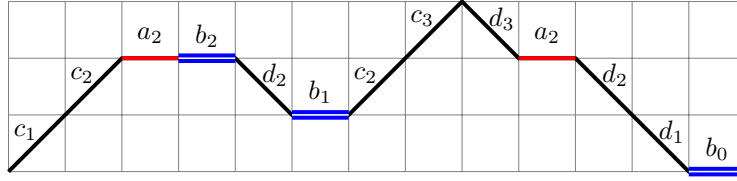


FIGURE 3. A 2-Motzkin path p in $\text{Motz}_2^{\leq 3}$ with $\text{wt}(p; a, b, c, d) = a_2^2 b_0 b_1 b_2 c_1 c_2^2 c_3 d_1 d_2^2 d_3$. The blue horizontal edges are represented by double edges.

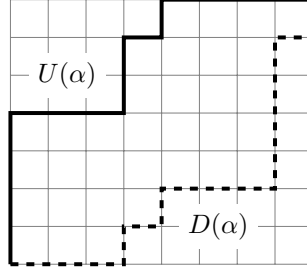


FIGURE 4. The boundary paths $U(\alpha)$ and $D(\alpha)$ for the connected skew shape $\alpha = (8, 7, 7, 7, 4, 3)/(4, 3, 3)$.

For sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$, define the *weight* $\text{wt}(p; a, b, c, d)$ of a 2-Motzkin path p to be the product of a_n (resp. b_n , c_n , and d_n) for each red horizontal step (resp. blue horizontal step, up step, and down step) starting at height n , see Figure 3.

Flajolet's theory [9] proves the following lemma.

Lemma 5.3. *Given sequences a, b, c, d , we have*

$$\sum_{p \in \text{Motz}_2^{\leq m}} \text{wt}(p; a, b, c, d) = \frac{1}{1 - a_0 - b_0 - \frac{c_0 d_1}{1 - a_1 - b_1 - \dots - \frac{c_{m-1} d_m}{1 - a_m - b_m}}}.$$

Observe that a connected skew shape $\alpha \in \text{CS}$ is determined by the two boundary paths starting from the bottom-left point to the top-right point consisting of north steps and east steps, which form the boundary of α . Let $U(\alpha)$ be the upper boundary path and $D(\alpha)$ the lower boundary path, see Figure 4.

There is a well known bijection between 2-Motzkin paths and connected skew shapes.

Definition 5.4 (The map $\phi : \text{Motz}_2^{\leq m} \rightarrow \text{CS}^{\leq m}$). Let $p \in \text{Motz}_2$. Then $\phi(p) = \alpha$ is the connected skew shape whose upper and lower boundary paths u, d are constructed by the following algorithm.

- (1) The first step of u (resp. d) is a north (resp. east) step.
- (2) For $i = 1, 2, \dots, n$, where n is the number of steps in p , the $(i + 1)$ st steps of u and d are defined as follows.
 - (a) If the i th step of p is an up step, then the $(i + 1)$ st step of u (resp. d) is a north (resp. east) step.

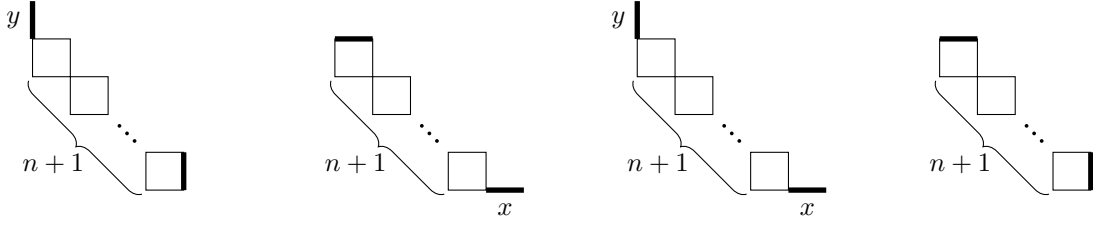


FIGURE 5. From left to right are shown the pairs of steps in $U(\alpha)$ and $D(\alpha)$ corresponding to a red horizontal step, a blue horizontal step, an up step, and a down step starting at height n in p when $\phi(p) = \alpha$. The number of squares whose centers are on the line connecting the starting points of the steps in $U(\alpha)$ and $D(\alpha)$ is $n + 1$. In each diagram an x (resp. y) is written when a new column (resp. row) is created.

- (b) If the i th step of p is a down step, then the $(i + 1)$ st step of u (resp. d) is a east (resp. north) step.
 - (c) If the i th step of p is a red horizontal step, then the $(i + 1)$ st steps of u and d are both north steps.
 - (d) If the i th step of p is a blue horizontal step, then the $(i + 1)$ st steps of u and d are both east steps.
- (3) Finally, the last (the $(n + 2)$ nd) step of u (resp. d) is an east (resp. north) step.

For example, if p is the 2-Motzkin path in Figure 3, then $\phi(p)$ is the connected skew shape α in Figure 4.

Proposition 5.5. *The map $\phi : \text{Motz}_2^{\leq m} \rightarrow \text{CS}^{\leq m}$ is a bijection. Moreover, if a, b, c , and d are the sequences given by $a_n = q^{n+1}y$, $b_n = q^{n+1}x$, $c_n = q^{n+1}xy$, and $d_n = q^{n+1}$, and if $\phi(p) = \alpha$, then*

$$x^{\text{col}(\alpha)} y^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = qxy \cdot \text{wt}(p; a, b, c, d).$$

Proof. One can easily prove this by the observation that if the i th step of p starts at height n , then the number of squares whose centers are on the line connecting the starting points of the $(i + 1)$ st steps of $U(\alpha)$ and $D(\alpha)$ is $n + 1$, see Figure 5. We omit the details. \square

Proposition 1.8 in the introduction follows from Lemma 5.3 and Proposition 5.5. Now we can easily prove Theorem 1.9 in the introduction. Let us state the theorem again.

Theorem 5.6. *For any integer $m \geq 1$, we have*

$$\sum_{\alpha \in \text{CS}^{\leq m}} (q^\nu z^2)^{\text{col}(\alpha)} (q^\nu)^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = -q^\nu z \frac{R_{m, \nu+2}(z; q^{-1})}{R_{m+1, \nu+1}(z; q^{-1})}.$$

Proof. Let $x = q^\nu z^2$ and $y = q^\nu$. Then by Proposition 1.8 the left hand side of the equation is

$$(5.4) \quad \sum_{\alpha \in \text{CS}^{\leq m}} x^{\text{col}(\alpha)} y^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = \frac{qxy}{1 - q(x+y) - \frac{q^3xy}{1 - q^2(x+y) - \dots - \frac{q^{2m+1}xy}{1 - q^{m+1}(x+y)}}}.$$

Using Theorem 5.1, Definition 4.5, and Lemma 4.2 with $c_i = 1 - q^{\nu+i+1}$, we obtain that the right hand side of the equation is

$$\begin{aligned} -q^\nu z \frac{R_{m,\nu+2}(z; q^{-1})}{R_{m+1,\nu+1}(z; q^{-1})} &= \frac{q^{2\nu+1} z^2}{1 - q^{\nu+1}} \sum_{n \geq 0} \mu_n^{\leq m}(b, a, \lambda) z^{2n} \\ &= -(1 - q^\nu) \mathbf{K}_{i=0}^m \left(\frac{-q^{2\nu+2i+1} z^2 / (1 - q^{\nu+i})(1 - q^{\nu+i+1})}{1 - q^{\nu+i+1} z^2 / (1 - q^{\nu+i+1})} \right) \\ &= -\mathbf{K}_{i=0}^m \left(\frac{-q^{2\nu+2i+1} z^2}{1 - q^{\nu+i+1} - q^{\nu+i+1} z^2} \right) = -\mathbf{K}_{i=0}^m \left(\frac{-q^{2i+1} xy}{1 - q^{i+1}(x+y)} \right), \end{aligned}$$

which is the same as the right hand side of (5.4). \square

6. q -NÖRLUND AND HEINE CONTINUED FRACTIONS

In the previous section we have shown that the ratio $J_{\nu+1}(z; q^{-1})/J_\nu(z; q^{-1})$ is a generating function for moments of orthogonal polynomials of type R_I . In this section we show that this ratio can also be written a generating function for moments of usual orthogonal polynomials.

Recall from (1.13) that

$$\frac{J_{\nu+1}(z; q^{-1})}{J_\nu(z; q^{-1})} = \frac{-q^{\nu+1} z}{1 - q^{\nu+1}} \cdot \frac{{}_1\phi_1(0; q^{\nu+2}; q, q^{\nu+2} z^2)}{{}_1\phi_1(0; q^{\nu+1}; q, q^{\nu+1} z^2)}.$$

Our strategy is to find two continued fraction expressions for the ratio of ${}_1\phi_1$'s in the above equation using the q -Nörlund continued fraction and Heine's continued fraction.

First we state the q -Nörlund fraction [5, (19.2.7)].

Lemma 6.1 (q -Nörlund fraction). *We have*

$$\frac{{}_2\phi_1(a, b; c; q, z)}{{}_2\phi_1(aq, bq; cq; q, z)} = \frac{1 - c - (a + b - ab - abq)z}{1 - c} + \frac{1}{1 - c} \mathbf{K}_{m=1}^\infty \left(\frac{c_m(z)}{e_m + d_m z} \right),$$

where

$$\begin{aligned} c_m(z) &= (1 - aq^m)(1 - bq^m)(cz - abq^m z^2)q^{m-1}, \\ e_m &= 1 - cq^m, \\ d_m &= -(a + b - abq^m - abq^{m+1})q^m. \end{aligned}$$

The q -Nörlund fraction can be restated in the form of a continued fraction for type R_I orthogonal polynomials.

Proposition 6.2 (q -Nörlund fraction restated). *We have*

$$\frac{{}_2\phi_1(aq, bq; cq; q, z)}{{}_2\phi_1(a, b; c; q, z)} = \frac{1}{1 - b_0 z - \frac{a_1 z + \lambda_1 z^2}{1 - b_1 z - \frac{a_2 z + \lambda_2 z^2}{1 - b_2 z - \dots}}},$$

where

$$\begin{aligned} b_m &= \frac{(a + b - abq^m - abq^{m+1})q^m}{1 - cq^m}, \\ a_m &= -\frac{(1 - aq^m)(1 - bq^m)cq^{m-1}}{(1 - cq^{m-1})(1 - cq^m)}, \\ \lambda_m &= \frac{(1 - aq^m)(1 - bq^m)abq^{2m-1}}{(1 - cq^{m-1})(1 - cq^m)}. \end{aligned}$$

Proof. By taking the inverse on each side of the equation in Lemma 6.1 we obtain

$$(6.1) \quad \frac{{}_2\phi_1(aq, bq; cq; q, z)}{{}_2\phi_1(a, b; c; q, z)} = \frac{1-c}{c_0(z)} \mathbf{K}_{m=0}^{\infty} \left(\frac{c_m(z)}{e_m + d_m z} \right).$$

Applying Lemma 4.2 with $c_i = 1/(1 - cq^i)$ and $m \rightarrow \infty$ yields

$$\frac{{}_2\phi_1(aq, bq; cq; q, z)}{{}_2\phi_1(a, b; c; q, z)} = \frac{(1 - cq^{-1})(1 - c)}{c_0(z)} \mathbf{K}_{m=0}^{\infty} \left(\frac{c_m(z)/(1 - cq^{m-1})(1 - cq^m)}{e_m/(1 - cq^m) + d_m z/(1 - cq^m)} \right),$$

which is the same as the desired identity. \square

Heine's contiguous relation [7, 17.6.19] is

$${}_2\phi_1(aq, b; cq; q, z) - {}_2\phi_1(a, b; c; q, z) = \frac{(1-b)(a-c)z}{(1-c)(1-cq)} {}_2\phi_1(aq, bq; cq^2; q, z).$$

Equivalently,

$$(6.2) \quad \frac{{}_2\phi_1(aq, b; cq; q, z)}{{}_2\phi_1(a, b; c; q, z)} = \frac{1}{1 - \frac{(1-b)(a-c)z}{(1-c)(1-cq)} \cdot \frac{{}_2\phi_1(bq, aq; cq^2; q, z)}{{}_2\phi_1(b, aq; cq; q, z)}}.$$

Applying (6.2) iteratively, we obtain Heine's continued fraction, which is a q -analog of Gauss's continued fraction.

Lemma 6.3 (Heine's fraction). *We have*

$$\frac{{}_2\phi_1(aq, b; cq; q, z)}{{}_2\phi_1(a, b; c; q, z)} = \frac{1}{1 - \frac{\beta_1 z}{1 - \frac{\beta_2 z}{1 - \dots}}},$$

where

$$\beta_{2n+1} = \frac{(1 - bq^n)(a - cq^n)q^n}{(1 - cq^{2n})(1 - cq^{2n+1})}, \quad \beta_{2n} = \frac{(1 - aq^n)(b - cq^n)q^{n-1}}{(1 - cq^{2n-1})(1 - cq^{2n})}.$$

Now we give two different continued fraction expressions for a ratio of ${}_1\phi_1$'s.

Proposition 6.4. *We have*

$$(6.3) \quad \frac{{}_1\phi_1(0; cq; q, qz)}{{}_1\phi_1(0; c; q, z)} = \frac{1}{1 - b_0z - \frac{a_1z}{1 - b_1z - \frac{a_2z}{1 - b_2z - \dots}}}$$

$$(6.4) \quad = \frac{1}{1 - \frac{\lambda_1z}{1 - \frac{\lambda_2z}{1 - \dots}}},$$

where

$$a_i = \frac{cq^{2i-1}}{(1 - cq^{i-1})(1 - cq^i)}, \quad b_i = \frac{q^i}{1 - cq^i},$$

and

$$\lambda_{2i} = \frac{cq^{3i-1}}{(1 - cq^{2i-1})(1 - cq^{2i})}, \quad \lambda_{2i+1} = \frac{q^i}{(1 - cq^{2i})(1 - cq^{2i+1})}.$$

Proof. We use the well known fact [11, p.5]:

$$\lim_{a_1 \rightarrow \infty} {}_r\phi_s \left(a_1, \dots, a_r; b_1, \dots, b_s; q, \frac{z}{a_1} \right) = {}_{r-1}\phi_s (a_2, \dots, a_r; b_1, \dots, b_s; q, z).$$

Equation (6.3) (resp. Equation (6.4)) is obtained by replacing $b \mapsto 0$, $z \mapsto z/a$ and sending a to infinity in Proposition 6.2 (resp. Lemma 6.3). \square

Using Proposition 6.4 we obtain two different continued fraction expressions for the ratio $J_{\nu+1}(z; q^{-1})/J_{\nu}(z; q^{-1})$, one of which is given in Theorem 5.1.

Theorem 6.5. *Let $b, a, \lambda, b', a', \lambda'$ be the sequences given by*

$$b_n = \frac{q^{\nu+n+1}}{1 - q^{\nu+n+1}}, \quad a_n = \frac{q^{2\nu+2n+1}}{(1 - q^{\nu+n})(1 - q^{\nu+n+1})}, \quad \lambda_n = 0,$$

$$b'_n = a'_n = 0, \quad \lambda'_{2i} = \frac{q^{2\nu+3i+1}}{(1 - q^{\nu+2i})(1 - q^{\nu+2i+1})}, \quad \lambda'_{2i+1} = \frac{q^{\nu+i+1}}{(1 - q^{\nu+2i+1})(1 - q^{\nu+2i+2})}.$$

Then

$$(6.5) \quad \frac{J_{\nu+1}(z; q^{-1})}{J_{\nu}(z; q^{-1})} = \frac{q^{\nu+1}z}{q^{\nu+1} - 1} \sum_{n \geq 0} \mu_n(b, a, \lambda) z^{2n}$$

$$(6.6) \quad = \frac{q^{\nu+1}z}{q^{\nu+1} - 1} \sum_{n \geq 0} \mu_{2n}(b', a', \lambda') z^{2n}.$$

Proof. The first identity (6.5) has been already proved in Theorem 5.1. Alternatively, applying (6.3) to (1.13) gives another proof of (6.5). Applying (6.4) to (1.13) gives the second identity (6.6). \square

Note that the above theorem shows that $J_{\nu+1}(z; q^{-1})/J_{\nu}(z; q^{-1})$ is (up to a scalar multiplication) the moment generating function for both orthogonal polynomials of type R_I and usual orthogonal polynomials. Using Flajolet's theory on continued fractions one obtains two combinatorial interpretations for the coefficients in the series expansion of this ratio. Note also that we get a similar result for the ratio $J_{\nu+1}(z; q)/J_{\nu}(z; q)$ if we replace q by q^{-1} in Theorem 6.5.

Remark 6.6. By replacing $a \mapsto 0$, $x \mapsto x/b$ and sending b to infinity in Lemma 6.3 we obtain

$$(6.7) \quad \frac{{}_1\phi_1(0; cq; q, z)}{{}_1\phi_1(0; c; q, z)} = \frac{1}{1 - \frac{\lambda_1 z}{1 - \frac{\lambda_2 z}{1 - \dots}}},$$

where

$$\lambda_{2n} = \frac{q^{n-1}}{(1 - cq^{2n-1})(1 - cq^{2n})}, \quad \lambda_{2n+1} = \frac{cq^{3n}}{(1 - cq^{2n})(1 - cq^{2n+1})}.$$

One can easily check that (6.7) is equivalent to (6.4) using the fact

$${}_1\phi_1(0; c; q, z) = {}_1\phi_1(0; c^{-1}; q^{-1}, q^{-1}c^{-1}z),$$

which implies

$$\frac{{}_1\phi_1(0; qc; q, qz)}{{}_1\phi_1(0; c; q, z)} = \frac{{}_1\phi_1(0; q^{-1}c^{-1}; q^{-1}, q^{-1}c^{-1}z)}{{}_1\phi_1(0; c^{-1}; q^{-1}, q^{-1}c^{-1}z)}.$$

Remark 6.7. Recall (1.1) that the ratio $J_{1/2}(z)/J_{-1/2}(z)$ is the tangent function. The *tangent numbers* or (*odd*) *Euler numbers* E_{2n+1} are defined by

$$\tan x = \sum_{n=0}^{\infty} \frac{E_{2n+1} x^{2n+1}}{(2n+1)!}.$$

The following q -analog of tangent numbers has been studied in several papers [10, 12, 13, 24, 26]:

$$\sum_{n=0}^{\infty} \frac{E_{2n+1}^*(q) x^{2n+1}}{(q; q)_{2n+1}} = \frac{x}{1-q} \cdot \frac{{}_1\phi_1(0; q^3; q^2, q^2 x^2)}{{}_1\phi_1(0; q; q^2, x^2)} = -q^{1/2} \frac{J_{1/2}(q^{-1/2}x; q^{-2})}{J_{-1/2}(q^{-1/2}x; q^{-2})},$$

where the last equality follows from (1.13). Hwang et al. [13, (30),(31)] found two continued fractions for this generating function, which are both special cases of Theorem 6.5. There are combinatorial objects related to this generating function such as alternating permutations and skew semistandard Young tableaux [13, 24, 27]. It would be interesting to generalize these results for an arbitrary ν .

The ratio of two ${}_1\phi_1$'s in Proposition 6.4 is related to $J_{\nu+1}(z; q^{-1})/J_{\nu}(z; q^{-1})$. Similarly, we can consider the ratio corresponding to

$$(6.8) \quad \frac{J_{\nu+1}^{(1)}(z; q)}{J_{\nu}^{(1)}(z; q)} = \frac{z/2} {1 - q^{\nu+1}} \frac{{}_2\phi_1(0, 0; q^{\nu+2}; q, -z^2/4)}{{}_2\phi_1(0, 0; q^{\nu+1}; q, -z^2/4)},$$

where $J_{\nu}^{(1)}(z; q)$ is Jackson's q -Bessel function defined in (3.1). In this case, both Heine's fraction (Lemma 6.3) and the q -Nörlund fraction (Proposition 6.2) with $a = b = 0$ give the same continued fraction:

$$(6.9) \quad \frac{{}_2\phi_1(0, 0; cq; q, z)}{{}_2\phi_1(0, 0; c; q, z)} = \frac{1}{1 - \frac{a_1 z}{1 - \frac{a_2 z}{1 - \dots}}},$$

where

$$a_n = \frac{-cq^{n-1}}{(1 - cq^{n-1})(1 - cq^n)}.$$

Applying (6.9) to (6.8) we obtain the following theorem.

Theorem 6.8. *Let b, a, λ be the sequences given by*

$$b_n = \lambda_n = 0, \quad a_n = \frac{q^{\nu+n}}{(1-q^{\nu+n})(1-q^{\nu+n+1})}.$$

Then

$$\frac{J_{\nu+1}^{(1)}(z; q)}{J_{\nu}^{(1)}(z; q)} = \frac{z/2}{1-q^{\nu+1}} \sum_{n \geq 0} \mu_n(b, a, \lambda) \left(-\frac{z^2}{4}\right)^n.$$

Equivalently, letting b', a', λ' be the sequences given by

$$b'_n = a'_n = 0, \quad \lambda'_n = \frac{q^{\nu+n}}{(1-q^{\nu+n})(1-q^{\nu+n+1})},$$

we have

$$\frac{J_{\nu+1}^{(1)}(z; q)}{J_{\nu}^{(1)}(z; q)} = \frac{z/2}{1-q^{\nu+1}} \sum_{n \geq 0} \mu_n(b', a', \lambda') \left(\frac{iz}{2}\right)^n.$$

Note that as before the above theorem shows that $J_{\nu+1}^{(1)}(z; q)/J_{\nu}^{(1)}(z; q)$ is (up to a scalar multiplication) the moment generating function for orthogonal polynomials, which gives a combinatorial interpretation for the coefficients in the series expansion of this ratio.

7. OPEN PROBLEMS

Recall that Kishore's theorem is a statement about the power series coefficients of the ratio $J_{\nu+1}(x)/J_{\nu}(x)$ of two Bessel functions. We conjecture the following finite version of Kishore's theorem on a ratio of Lommel polynomials $R_{m,\nu}(x)$ defined in the introduction.

Conjecture 7.1. Let

$$\frac{R_{m,\nu+2}(x)}{R_{m+1,\nu+1}(x)} = \sum_{n=0}^{\infty} \frac{N_{n,\nu}^{(m)}}{D_{n,\nu}^{(m)}} \left(\frac{x}{2}\right)^{2n+1},$$

where

$$D_{n,\nu}^{(m)} = \prod_{k=0}^m (\nu+k+1)^{f(m,n,k)},$$

$$f(m,n,k) = \begin{cases} \max\left(\left\lfloor \frac{n+1}{k+1} \right\rfloor, \left\lfloor \frac{n+m-2k+1}{m-k+1} \right\rfloor\right), & \text{if } k \neq m/2, \\ 1, & \text{if } k = m/2. \end{cases}$$

Then $N_{n,\nu}^{(m)}$ is a polynomial in ν with nonnegative integer coefficients.

By (1.8) Conjecture 7.1 implies Kishore's Theorem.

In Section 6 we saw that the ratio

$$\frac{J_{\nu+1}(z; q^{-1})}{J_{\nu}(z; q^{-1})} = \frac{-q^{\nu+1}z}{1-q^{\nu+1}} \cdot \frac{{}_1\phi_1(0; q^{\nu+2}; q, q^{\nu+2}z^2)}{{}_1\phi_1(0; q^{\nu+1}; q, q^{\nu+1}z^2)}$$

has two generalizations, the q -Nörlund continued fraction and Heine's continued fraction. These two generalizations seem to have a similar property as follows.

Conjecture 7.2. Let

$$\sum_{n \geq 0} \gamma_n(a, b, c) z^n = \frac{{}_2\phi_1(aq, bq; cq; q, z)}{{}_2\phi_1(a, b; c; q, z)}.$$

Then

$$\frac{\gamma_n(a, b, c)}{1 - c} = \frac{P_n(a, b, c)}{\prod_{k=0}^n (1 - cq^k)^{\lfloor \frac{n+1}{k+1} \rfloor}},$$

for some polynomial $P_n(a, b, c)$ in a, b, c, q with integer coefficients.

Conjecture 7.3. Let

$$\sum_{n \geq 0} \gamma'_n(a, b, c) z^n = \frac{{}_2\phi_1(aq, b; cq; q, z)}{{}_2\phi_1(a, b; c; q, z)}.$$

Then

$$\frac{\gamma'_n(a, b, c)}{1 - c} = \frac{P'_n(a, b, c)}{\prod_{k=0}^n (1 - cq^k)^{\lfloor \frac{n+1}{k+1} \rfloor}},$$

for some polynomial $P'_n(a, b, c)$ in a, b, c, q with integer coefficients.

Using Flajolet's theory one can reinterpret the equality of the two continued fractions in Proposition 6.4 completely combinatorially.

Problem 7.4. Find a combinatorial proof of the identity

$$\frac{1}{1 - b_0 z - \frac{a_1 z}{1 - b_1 z - \frac{a_2 z}{1 - b_2 z - \dots}}} = \frac{1}{1 - \frac{\lambda_1 z}{1 - \frac{\lambda_2 z}{1 - \dots}}},$$

where

$$a_i = \frac{cq^{2i-1}}{(1 - cq^{i-1})(1 - cq^i)}, \quad b_i = \frac{q^i}{1 - cq^i},$$

and

$$\lambda_{2i} = \frac{cq^{3i-1}}{(1 - cq^{2i-1})(1 - cq^{2i})}, \quad \lambda_{2i+1} = \frac{q^i}{(1 - cq^{2i})(1 - cq^{2i+1})}.$$

Finally we remark that more general ratios of hypergeometric series are considered in [25]. It would be interesting to see whether the results in this paper can be extended to these ratios.

ACKNOWLEDGMENTS

The authors would like to thank Xavier Viennot for references on combinatorics of q -Bessel functions, and Mourad Ismail for information on the third q -Bessel function.

REFERENCES

- [1] J.-C. Aval, A. Boussicault, M. Bouvel, and M. Silimbani. Combinatorics of non-ambiguous trees. *Advances in Applied Mathematics*, 56:78–108, 2014.
- [2] E. Barucci, A. Del Lungo, J. M. Fédou, and R. Pinzani. Steep polyominoes, q -Motzkin numbers and q -Bessel functions. *Discrete Math.*, 189(1-3):21–42, 1998.
- [3] E. Barucci, A. Lungo, E. Pergola, and R. Pinzani. Some permutations with forbidden subsequences and their inversion number. *Discrete Mathematics*, 234(1-3):1–15, 2001.
- [4] M. Bousquet-Mélou and X. G. Viennot. Empilements de segments et q -énumération de polyominos convexes dirigés. *Journal of Combinatorial Theory, Series A*, 60(2):196–224, 1992.
- [5] A. Cuyt, V. B. Petersen, B. Verdonk, H. Waadeland, and W. B. Jones. *Handbook of continued fractions for special functions*. Springer, New York, 2008.
- [6] M.-P. Delest and J.-M. Fédou. Enumeration of skew Ferrers diagrams. *Discrete Math.*, 112(1-3):65–79, 1993.
- [7] *NIST Digital Library of Mathematical Functions*. <http://dlmf.nist.gov/>, Release 1.0.26 of 2020-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

- [8] J. Fédou. Combinatorial objects enumerated by q -Bessel functions. *Reports on Mathematical Physics*, 34(1):57–70, 1994.
- [9] P. Flajolet. Combinatorial aspects of continued fractions. *Discrete Math.*, 32(2):125–161, 1980.
- [10] M. Fulmek. A continued fraction expansion for a q -tangent function. *Sém. Lothar. Combin.*, 45:Art. B45b, 5, 2000/01.
- [11] G. Gasper and M. Rahman. *Basic hypergeometric series*, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Richard Askey.
- [12] T. Huber and A. J. Yee. Combinatorics of generalized q -Euler numbers. *J. Combin. Theory Ser. A*, 117(4):361–388, 2010.
- [13] B.-H. Hwang, J. S. Kim, M. Yoo, and S.-m. Yun. Reverse plane partitions of skew staircase shapes and q -Euler numbers. *J. Combin. Theory Ser. A*, 168:120–163, 2019.
- [14] M. E. H. Ismail. *Classical and quantum orthogonal polynomials in one variable*, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2009.
- [15] M. E. H. Ismail and D. R. Masson. Generalized orthogonality and continued fractions. *J. Approx. Theory*, 83(1):1–40, 1995.
- [16] E. Y. Jin. Heaps and two exponential structures. *European Journal of Combinatorics*, 54:87–102, 2016.
- [17] W. B. Jones and W. Thron. Survey of continued fraction methods of solving moment problems and related topics. In *Analytic theory of continued fractions*, pages 4–37. Springer, 1982.
- [18] S. Kamioka. Laurent biorthogonal polynomials, q -Narayana polynomials and domino tilings of the Aztec diamonds. *J. Combin. Theory Ser. A*, 123:14–29, 2014.
- [19] J. S. Kim and D. Stanton. Combinatorics of orthogonal polynomials of type R_I . in preparation.
- [20] N. Kishore. The Rayleigh polynomial. *Proc. Amer. Math. Soc.*, 15:911–917, 1964.
- [21] H. Koelink and R. Swarttouw. On the zeros of the Hahn-Exton q -Bessel function and associated q -Lommel polynomials. *Journal of Mathematical Analysis and Applications*, 186(3):690–710, 1994.
- [22] D. H. Lehmer. Zeros of the Bessel Function $J_n(x)$. *Math. Comp.*, 1:405–407, 1945.
- [23] Y. Li. A q -analogue and a symmetric function analogue of a result by Carlitz, Scoville and Vaughan. *Preprint*, arXiv:1811.06180v1.
- [24] A. H. Morales, I. Pak, and G. Panova. Hook formulas for skew shapes II. Combinatorial proofs and enumerative applications. *SIAM J. Discrete Math.*, 31(3):1953–1989, 2017.
- [25] M. Pétréolle, A. D. Sokal, and B.-X. Zhu. Lattice paths and branched continued fractions: An infinite sequence of generalizations of the Stieltjes–Rogers and Thron–Rogers polynomials, with coefficientwise Hankel-total positivity. <https://arxiv.org/abs/1807.03271>.
- [26] H. Prodinger. A continued fraction expansion for a q -tangent function: an elementary proof. *Sém. Lothar. Combin.*, 60:Art. B60b, 3, 2008/09.
- [27] R. P. Stanley. A survey of alternating permutations. In *Combinatorics and graphs*, volume 531 of *Contemp. Math.*, pages 165–196. Amer. Math. Soc., Providence, RI, 2010.
- [28] G. Viennot. Une théorie combinatoire des polynômes orthogonaux. Lecture Notes, UQAM, 1983.
- [29] G. N. Watson. *A treatise on the theory of Bessel functions, second edition*. Cambridge University Press, 1944.

DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON, SOUTH KOREA
Email address: jangsookim@skku.edu

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA, USA
Email address: stanton@math.umn.edu