1. (30 points) **Consider the surface parametrized by** $(x, y, z) = \Phi(x, y) = (x, y, 4-(x^2+y^2))$ **between the planes** $z = 1$ **and** $z = 3$**.**

   (i) (15 points) **Set up the integral to find the surface area.**

   Call the Surface S and let $g(x, y) = 4 - (x^2 + y^2)$. Then S is parametrized by $\Phi(x, y) = (x, y, g(x, y))$ for $(x, y) \in D$ where D is the annulus with outer radius of $\sqrt{3}$ and inner radius of 1. We have:

$$A(S) = \int\int_D \sqrt{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2 + 1}\, dA$$
$$= \int\int_D \sqrt{(-2x)^2 + -(2y)^2 + 1}\, dA$$
$$= \int\int_D \sqrt{4x^2 + 4y^2 + 1}\, dA$$

   (ii) (10 points) **In the resulting double integral, change variables to polar coordinates.**

   Letting $x = r\ cos\theta$ and $y = r\ sin\theta$, we find

$$A(S) = \int\int_D \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi} \int_1^{\sqrt{3}} \sqrt{4r^2 + 1}\ r\ drd\theta$$

   (iii) (5 points) **This integral should be easy to evaluate. Do it.**

   We use u substitution. Let $u = 4r^2 + 1$. Then $du = 8r\ dr$, so

   $\int_0^{2\pi} \int_1^{\sqrt{3}} \sqrt{4r^2 + 1}\ r\ drd\theta = \frac{1}{8} \int_0^{2\pi} \int_5^{13} \sqrt{u}\ dud\theta$

   $= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} u^{\frac{3}{2}} |_5^{13}$

   $= \int_0^{2\pi} \frac{1}{12} (13\sqrt{13} - 5\sqrt{5})$

   $= (-\frac{2\pi}{12})(13\sqrt{13} - 5\sqrt{5})$

   $= \frac{\pi}{6}(13\sqrt{13} - 5\sqrt{5})$

2. (20 points) **Compute the integral** $\iint_S \mathbf{F} \cdot d\mathbf{S}$**, where** $\mathbf{F}(x, y, z) = (y - z, x - z, x - y)$ **and** S **is the planar surface parametrized by** $\mathbf{\Phi}(u, v) = (u - v, u + v, u)$ **for** $0 \le u \le 1$ **and** $0 \le v \le 1$**. Orient the surface so the first component of the normal vector is positive.**

   Since we are given the parametrization of the planar surface by $\mathbf{\Phi}$, the surface integral of $\mathbf{F}$ over the surface S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{\Phi}(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv \quad \text{(3 points)}$$

First we find

$$\mathbf{T}_u = \mathbf{i}+\mathbf{j}+\mathbf{k} \quad \text{and} \quad \mathbf{T}_v = -\mathbf{i}+\mathbf{j}+0\mathbf{k} \quad \text{(2 points each)}$$

and hence the normal vector is

$$\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = -\mathbf{i}-\mathbf{j}+2\mathbf{k}\,. \quad \text{(4 points)}$$

However since the orientation is given so that the first component of the normal vector should be positive we choose the normal vector as

$$\mathbf{N} = \mathbf{T}_v \times \mathbf{T}_u = \mathbf{i}+\mathbf{j}-2\mathbf{k}\,. \quad \text{(right orientation 2 points)}$$

Then we also find

$$\mathbf{F}(\mathbf{\Phi}(u,v)) = (u+v-u)\mathbf{i} + (u-v-u)\mathbf{j} + (u-v-u-v)\mathbf{k} = v\mathbf{i} - v\mathbf{j} - 2v\mathbf{k}\,. \quad \text{(2 points)}$$

Thus,

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{\Phi}(u,v))\cdot(\mathbf{T}_u\times\mathbf{T}_v)\,du\,dv \\
&= \int_0^1\int_0^1 (v\mathbf{i}-v\mathbf{j}-2v\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-2\mathbf{k})\,du\,dv \\
&= \int_0^1\int_0^1 (v-v+4v)\,dv\,du = \int_0^1\int_0^1 4v\,dv\,du \\
&= \int_0^1 \left(\cancelto{2}{4}\, \frac{v^2}{\cancel{2}}\Bigg|_0^1 \right) du = \int_0^1 2\,du = 2u|_0^1 = 2 \quad \text{(5 points)}
\end{aligned}$$

3. (20 points) **Let** $\mathbf{F}(x,y) = (y\cos x + 2xe^y, \sin x + x^2e^y + 5)$**.**

   (i) (5 points) **Verify that F is a conservative vector field.**
   If $\nabla\times\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is a conservative vector field and if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ then

$$\nabla\times\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

   Here $P(x,y) = y\cos x + 2xe^y$ and $Q(x,y) = \sin x + x^2e^y + 5$ and thus

$$\nabla\times\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \left[(\cos x + 2xe^y) - (\cos x + 2xe^y)\right]\mathbf{k} = \mathbf{0} \quad \text{(5 points)}$$

   I also accepted the solutions where the student showed the existence of a potential function and stated that for that potential function f, $\nabla f = \mathbf{F}$ and hence $\mathbf{F}$ is a conservative vector field or just showed that scalar curl of $\mathbf{F}$ is zero.

   (ii) (15 points) **Evaluate** $\int_C \mathbf{F}\cdot d\mathbf{s}$ **where** C **is any curve from (0,1) to** $(\pi/2, 2)$**.** Since in (i) we showed that $\mathbf{F}$ is a conservative vector field then $\mathbf{F}$ has a potential function f. There are two ways of finding this (worth 8 points):

(A)

$$\begin{aligned}
f(x,y) &= \int_0^x P(t,0)\,dt + \int_0^y Q(x,t)\,dt \\
&= \int_0^x 2t\,dt + \int_0^y \left(\sin x + x^2e^t + 5\right)\,dt \\
&= t^2\Big|_0^x + \left(t\sin x + x^2e^t + 5t\right)\Big|_0^y \\
&= x^2 + y\sin x + x^2e^y + 5y - x^2 \\
&= y\sin x + x^2e^y + 5y
\end{aligned}$$

(B)

$$f(x,y) = \int P(x,y)\,dx = \int \left(y\cos x + 2xe^y\right)\,dx = y\sin x + x^2e^y + g(y)$$

Let's use the fact that $f_y(x,y) = Q(x,y)$.

$$f_y(x,y) = \sin x + x^2e^y + g'(y) = \sin x + x^2e^y + 5 \tag{1}$$

From Equation (1) we see that $g'(y) = 5$ and hence $g(y) = 5y$. Therefore $f(x,y) = y\sin x + x^2e^y + 5y$.

Since $\mathbf{F}$ is a conservative vector field it is path independent and hence

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{s} &= f(\mathbf{c}(b)) - f(\mathbf{c}(a)) = f(\pi/2, 2) - f(0,1) \\
&= 2\cancelto{1}{\sin\left(\frac{\pi}{2}\right)} + \frac{\pi^2}{4}e^2 + 10 - (1\cancelto{0}{\sin(0)} + 0 + 5) \\
&= 12 + \frac{pi^2}{4}e^2 - 5 = 7 + \frac{\pi^2}{4}e^2 \quad \text{(7 points)}
\end{aligned}$$

If a student chose to find a specific curve instead of finding a potential function 5 points were taken off and then if the line integral wasn't calculated properly another 5 points were taken off. If the parametrization of the curve was wrong no points were given.

4. (20 points) **Consider the integral** $\iint_D (4x^2 + 9y^2)dA$ **where** D **is the region bounded by the curve** $4x^2 + 9y^2 = 36$**.**

   (i) (10 points) **Let T be the transformation from a region** D^* **to** D **defined by** $(x,y) = \mathbf{T}(u,v) = (u/2, v/3)$**. Draw both the regions** D **and** D^***.**

   The region D is an ellipse centered at the origin where the radius in the x direction was 3 and the radius in the y direction was 2.

   The change of variables compressed in the x direction by a factor of 2 and compressed in the y direction by a factor of 3. Therefore, to go back to u and v coordinates, we need to stretch by a factor of 2 and 3 in the u and v directions, respectively. Hence the region D^* is a circle centered at the origin of radius 6.

   2 points off for having D^* be a unit circle. 5 points off if $D*$ was rectangle or triangle. 5 points off if D was a rectangle or triangle.

(ii) (10 points) **Change variables to an integral in u and v. (You need not evaluate the final integral.)**

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \frac{1}{6}$$
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{6}\right| = \frac{1}{6}$$

The integrand is $4x^2 + 9y^2 = u^2 + v^2$.
The bounds on the circle are, for example, $-6 \le v \le 6$, $-\sqrt{36 - v^2}u \le \sqrt{36 - v^2}$ so tha t the integral is

$$\iint_D (4x^2 + 9y^2) dA = \frac{1}{6}\int_{-6}^{6}\int_{-\sqrt{36-v^2}}^{\sqrt{36-v^2}} (u^2 + v^2) du\, dv$$

Getting $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ was worth 4 points, getting the integral was worth 2 points, getting the bounds was worth 4 points.

If your bounds matched your region drawn in part (i), you should have gotten the 4 points for the bounds even if the region was incorrect.

5. (20 points) **Let C be the circle parameterized by $(x, y, z) = (\cos t, \sin t, 4)$ for $0 \le t \le 2\pi$. Use Stokes' Theorem to calculate the circulation of the vector field**

$$\mathbf{F}(x, y, z) = (x + y)\mathbf{i} - (x + y + 2z)\mathbf{j} + (5x - 8z)\mathbf{k}$$

around C, which is the integral $\int_C \mathbf{F} \cdot d\mathbf{s}$. Sketch the curve, your chosen surface, along with a normal vector to show the surface orientation.

Since you are given a line integral and told to use Stokes' theorem, you must compute a surface integral over some surface whose boundary was C. So the first step is to choose a surface. Note that C is a circle of radius 1 in the plane $z = 4$ (i.e., satisfies $x^2 + y^2 = 1$ and $z = 4$). Possible correct surfaces include a disk in the plane $z = 4$ where $x^2 + y^2 \le 1$ (the most common and the simplest), a cone $z = \sqrt{x^2 + y^2}/4$ with $z \le 4$, and a half-sphere such as $z = \sqrt{1 - x^2 - y^2} + 4$.

Note that a cylinder such as $x^2 + y^2 = 1$ for $0 \le z \le 4$ is **not** a correct answer, since the boundary of the cylinder also includes the circle $x^2 + y^2 = 1$ in the xy-plane (i.e., where $z = 0$). 5 points were deducted for choosing a cylinder where C was half of the boundary of the cylinder. (More points were deducted if C was not part of the boundary.)

Note that any sphere (such as $(z - 4)^2 + x^2 + y^2 = 1$) does not have any boundary as it is a closed surface. Yes, it does include the circle C as part of the sphere, but C is not its boundary. (Choosing either the top or bottom half of the cylinder would be OK.) The integral of curl $\mathbf{F}$ over any closed surface must be zero. 5 points were deducted for choosing a sphere that included C as part of the sphere. (More points were deducted if C was not part of the sphere.)

If you computed a line integral directly, you were not using Stokes' theorem as instructed (since you were given the line integral to start with). If you computed a line integral directly, you were awarded at most 8 points (if everything was done correctly).

The following solution is for choosing the disk $x^2 + y^2 \leq 1$ and $z = 4$. We can parameterize this disk by

$$\Phi(u, v) = (u\cos v, u\sin v, 4), \qquad 0 \leq u \leq 1,\ 0 \leq v \leq 2\pi$$

The partial derivatives are

$$\begin{aligned}\mathbf{T}_u &= \frac{\partial \Phi}{\partial u} = (\cos v, \sin v, 0)\\ \mathbf{T}_v &= \frac{\partial \Phi}{\partial v} = (-u\sin v, u\cos v, 0)\end{aligned}$$

so that a normal vector is

$$\mathbf{T}_u \times \mathbf{T}_v = (0, 0, u)$$

This is an upward pointing normal vector. Since C is CCW when viewed from the positive z-axis, using this normal vector will make C be a positvely oriented boundary.

To use Stokes' theorem, we need to compute $\operatorname{curl}\mathbf{F} = (2, -5, -2)$.

Then, putting everything together, we obtain

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} (2, -5, -2)\cdot(0, 0, u) dv\, du\\ &= \int_0^1 \int_0^{2\pi} -2u dv\, du = -4\pi \int_0^1 u du = -2\pi\end{aligned}$$

Parametrizing a correct surface was 4 points, finding the correct normal vector was 4 points, calculating $\operatorname{curl}\mathbf{F}$ was 3 points, setting up the integral correctly was 5 points, evaluting it was 2 points, and the sketch was 2 points.

If you dropped the minus sign of the second component of $\mathbf{F}$ so that the last component of the curl was 0, you lost at least three points since this made the calculation trivially zero.

6. (30 points) **Consider the following triple integral**

$$\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_0^{2-\sqrt{x^2+y^2}} 1\, dz\, dy\, dx$$

(i) (5 points) **Describe the solid for which we would be finding its volume with this integral.**

The solid is described by the inequalities

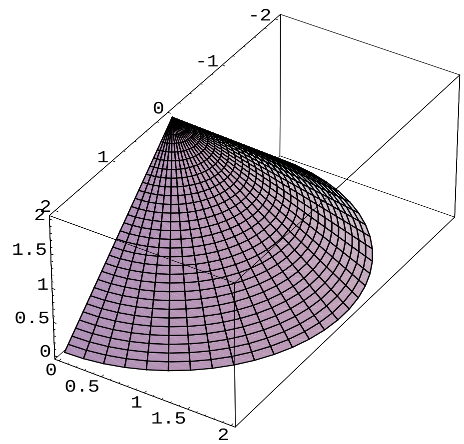
$$0 \leq z \leq 2 - \sqrt{x^2 + y^2} \tag{2}$$

$$0 \leq y \leq \sqrt{4 - x^2} \tag{3}$$

$$-2 \leq x \leq 2\,. \tag{4}$$

From (3) we have $y \geq 0$ and $y^2 \leq 4 - x^2$ i.e. $x^2 + y^2 \leq 4$ and $y \geq 0$. Notice that $x^2 + y^2 \leq 4$ is equivalent to $0 \leq 2 - \sqrt{x^2 + y^2}$. Therefore we can say that the solid lies above the plane $z = 0$, below the surface $z = 2 - \sqrt{x^2 + y^2}$ and $y \geq 0$. We know that $z = \pm\sqrt{x^2 + y^2}$ is the equation of the cone, thus $z = 2 - \sqrt{x^2 + y^2}$ is a bottom part of the cone shifted up by 2.

Putting everything together we see that the solid is bounded from bellow by the plane $z = 0$, from above by the cone $z = 2 - \sqrt{x^2 + y^2}$ and $y \geq 0$ i.e. we are considering half of the cone as shown on the figure.



(ii) (10 points) **Change variables in the integral to cylindrical coordinates.**
Let's introduce the cylindrical coordinates for $r \in [0, \infty), \theta \in [0, 2\pi), z \in (-\infty, \infty)$,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z\,.$$

We already showed that our system of inequalities is equivalent to $0 \leq z \leq 2 - \sqrt{x^2 + y^2}$, $y \geq 0$. In the cylindrical coordinates $0 \leq z \leq 2 - r$, $r\sin\theta \geq 0$ or $0 \leq z \leq 2 - r$ and $\theta \in [0, \pi]$. The answer in this case is then

$$\int_0^{\pi} \int_0^{2} \int_0^{2-r} r\ dz\, dr\, d\theta\,.$$

(iii) (15 points) **Change variables in the original integral to spherical coordinates.**
Let's introduce the spherical coordinates for $r \in [0, \infty), \theta \in [0, 2\pi), \phi \in [0, \pi)$,

$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi\,.$$

Then the inequalities $0 \leq z \leq 2 - \sqrt{x^2 + y^2}$, $y \geq 0$ transforms into

$$0 \leq r\cos\phi \leq 2 - \sqrt{(r\cos\theta\sin\phi)^2 + (r\sin\theta\sin\phi)^2} = 2 - r\sin\phi$$
$$0 \leq r\sin\theta\sin\phi\,.$$

From the second equation we have $\theta \in [0, \pi]$ and from the first one we have $r \leq \frac{2}{\cos\phi + \sin\phi}$ and $\phi \in [0, \frac{\pi}{2}]$. The answer in this case is then

$$\int_0^{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{\cos\phi+\sin\phi}} r^2 \sin\phi\ dr\, d\phi\, d\theta\,.$$