1. (20 points) For each statement, either give an example that shows the statement is TRUE, or give a brief reason that the statement is FALSE.

   (a) (10) There exists a pair of nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^3$ such that $\mathbf{v}\times\mathbf{w} = \mathbf{0}$ and $\mathbf{v}\cdot\mathbf{w} = 0$.

   FALSE: Since $0 = ||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\,||\mathbf{w}|| \sin\theta$ and $0 = \mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}|| \cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, we must have $\sin\theta$ and $\cos\theta$ both equal to 0, which is impossible.

   (b) (10) There exists a nonzero vector $\mathbf{v}$ in $\mathbf{R}^3$ such that $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||$.

   TRUE: Since $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2 = ||\mathbf{v}||$, we must have $||\mathbf{v}|| = 1$. Thus, any unit vector satisfies the condition; for example, $\mathbf{v} = (1, 0, 0)$.

2. (20 points) Suppose that $\mathbf{f}, \mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^2$ are functions such that

$$\mathbf{f}(u, v) = (uv, u^2 - v^2)$$

   and $\mathbf{g}(x, y) = (u(x, y), v(x, y))$ satisfies

$$\mathbf{g}(1, 2) = (5, -5)$$

   and

$$D\mathbf{g}(1, 2) = \begin{bmatrix} 2 & 4 \\ 6 & -8 \end{bmatrix}.$$

   (a) (10) Find $\frac{\partial v}{\partial x}(1, 2)$.

   $\frac{\partial v}{\partial x}(1, 2)$ is the $(2, 1)$-entry of $D\mathbf{g}(1, 2)$, which is equal to 6.

   (b) (10) Find $D(\mathbf{f} \circ \mathbf{g})(1, 2)$.

   By the chain rule, $D(\mathbf{f}\circ\mathbf{g})(1, 2) = D\mathbf{f}(\mathbf{g}(1, 2))\cdot D\mathbf{g}(1, 2)$. Since $\mathbf{g}(1, 2) = (5, -5)$ and $D\mathbf{f}(u, v) = \begin{bmatrix} v & u \\ 2u & -2v \end{bmatrix}$, we have

$$D(\mathbf{f} \circ \mathbf{g})(1, 2) = \begin{bmatrix} -5 & 5 \\ 10 & 10 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 6 & -8 \end{bmatrix} = \begin{bmatrix} 20 & -60 \\ 80 & -40 \end{bmatrix}.$$

3. (20 points) Let $z = f(x, y) = e^{-x}\sqrt{y}$.

   (a) (10) Find a function that is a linear approximation to $f(x, y)$ near the point $(0, 9)$. Write your linear approximation in the form $z = Ax + By + C$.

   $\nabla f = (-e^{-x}\sqrt{y}, \frac{1}{2}e^{-x}y^{-1/2})$ and so $\nabla f(0, 9) = (-3, \frac{1}{6})$. Also $f(0, 9) = 3$. The linear approximation is thus $z = 3 - 3(x - 0) + \frac{1}{6}(y - 6)$, or $z = -3x + \frac{1}{6}y + \frac{3}{2}$.

(b) (10) A calculator will tell you that $e^{-.01}\sqrt{9.6} \approx 3.067557214$. Estimate this quantity using the linear approximation in (a), and write your answer to the hundredths place. Show your work.

The linear approximation evaluated at (0.01, 9.6) is 3.07.

4. (25 points) Let $F(x, y, z) = x^2y + yz + z^2$.

(a) (15) Find $\nabla F(x, y, z)$.

$\nabla F(x, y, z) = (2xy, x^2 + z, y + 2z)$.

(b) (10) Are there any points (x, y, z) in upper-half-space (that is, where $z > 0$) at which the tangent plane to the level surface of F at that point is horizontal? Either find all such points, or give a reason why there are no such points.

NO: We would need $\nabla F(x, y, z)$ to have the form $(0, 0, c)$, which implies that $x^2 + z = 0$, which is impossible because $z > 0$.

5. (25 points) Let $g(x, y) = x + y^2$.

(a) (15) Find an equation for the plane that passes through the point $(3, -2, -4)$ and is parallel to the tangent plane to the graph of g at the point $(3, -2, 7)$. Write the equation in the form $Ax + By + Cz + D = 0$.

$\nabla g = (1, 2y)$, and so $\nabla g(3, -2) = (1, -4)$. The tangent plane is then given by $z = -4 + (x - 3) - 4(y + 2)$, or $x - 4y - z - 15 = 0$.

(b) (10) Sketch the level curve defined by the equation $g(x, y) = 5$ in the xy-plane. *Label the coordinate axes of your sketch.*

In the xy-plane with horizontal x-axis, the level curve is the parabola $x = 5 - y^2$, which opens to the left with x-intercept 5 and y-intercepts $\pm\sqrt{5}$.

6. (30 points) Let $f(x, y) = 10 - x^2 - 3y^2$.

(a) (10) Write a unit vector $\mathbf{u}$ that indicates the direction in which f is decreasing the fastest at the point $(2, 1)$.

$\nabla f = (-2x, -6y)$, and so $-\nabla f(2, 1) = (4, 6)$. Then $\mathbf{u} = \frac{1}{\sqrt{16+36}}(4, 6) = \frac{1}{\sqrt{13}}(2, 3)$.

(b) (10) Find the directional derivative of f at the point $(2, 1)$ in the direction $\mathbf{u}$ that you found in part (a).

The directional derivative of f in the direction $\mathbf{u}$ at the point $(2, 1)$ is equal to $\nabla f(2, 1) \cdot \mathbf{u} = -2\sqrt{13}$.

(c) (10) For each unit vector $\mathbf{v}$, let $\theta_{\mathbf{v}}$ denote the angle $0 \leq \theta_{\mathbf{v}} \leq \pi$ between $\mathbf{v}$ and the unit vector $\mathbf{u}$ that you found in (a). For which angles $\theta_{\mathbf{v}}$ is the directional derivative of f in the direction $\mathbf{v}$ at $(2,1)$ less than or equal to -5? Write your answer as an inequality in the form $A \leq \theta_{\mathbf{v}} \leq B$.

For each unit vector $\mathbf{v}$, the directional derivative of f in the direction $\mathbf{v}$ at the point $(2,1)$ is equal to $\nabla f(2,1) \cdot \mathbf{v} = ||(-4,-6)||\,||\mathbf{v}|| \cos\theta = 2\sqrt{13}\cos\theta$, where θ is the angle between ∇f and $\mathbf{v}$. So we need $2\sqrt{13}\cos\theta \leq -5$, or $\cos\theta \leq \frac{-5}{2\sqrt{13}}$. Thus $\cos^{-1}(\frac{-5}{2\sqrt{13}}) \leq \theta \leq \pi$, or $0 \leq \theta_{\mathbf{v}} \leq \pi - \cos^{-1}(\frac{-5}{2\sqrt{13}}) = \cos^{-1}(\frac{5}{2\sqrt{13}})$.