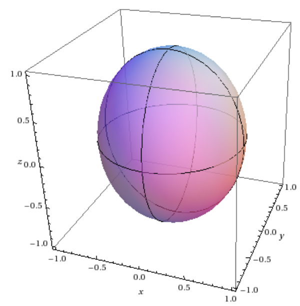
Third Midterm: Solutions

1. Rewrite the given equation in the form $\frac{x^2}{\frac{2}{3}} + \frac{y^2}{\frac{2}{3}} + \frac{z^2}{1} = 1$. This equation defines an ellipsoid centered at the origin with the x-, y- and z- radii equal to $\sqrt{\frac{2}{3}}$, $\sqrt{\frac{2}{3}}$ and 1 respectively. It looks as follows:



Such an ellipsoid can be parametrized by

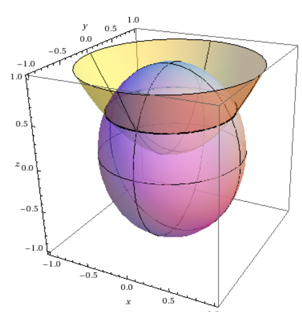
$$x(\theta,\phi) = \sqrt{\frac{2}{3}}\sin\phi\cos\theta, \quad y(\theta,\phi) = \sqrt{\frac{2}{3}}\sin\phi\sin\theta, \quad z(\theta,\phi) = \cos\phi. \tag{1}$$

It is important to notice here that the parameter ϕ is **not** the angle between a radius-vector and the z-axis.

The condition $z \geq x^2 + y^2$ puts certain restrictions on ϕ. To determine them, we find the intersection of the given ellipsoid and the surface[1] $z = x^2 + y^2$:

$$\begin{cases} 3(x^2+y^2) + 2z^2 = 2 \\ z = x^2 + y^2. \end{cases}$$

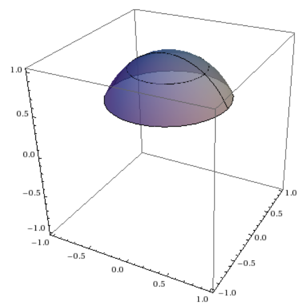
We plug $z = x^2 + y^2$ into the first equation and obtain $2z^2 + 3z + 2 = 0$. The positive solution of this quadratic equation is $z = \frac{1}{2}$. Now, we plug $z = \frac{1}{2}$ into the first equation of the system again and obtain $x^2 + y^2 = \frac{1}{2}$. Hence, the intersection is a circle of radius $\frac{1}{\sqrt{2}}$ lying on the plane $z = \frac{1}{2}$.



[1] It is a paraboloid.

If $z = \frac{1}{2}$, then, according to (1), $\phi = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$. Therefore, ϕ ranges from 0 to $\frac{\pi}{3}$ and θ goes from 0 to 2π.

This is how the surface looks like:



2. We denote the given vector field $6xy(\cos z)\mathbf{i}+3x^2(\cos z)\mathbf{j}-3x^2y(\sin z)\mathbf{k}$ by $\mathbf{F} = (F_1, F_2, F_3)$. All F_i's are continuously differentiable, and it is easy to verify that $\nabla \times \mathbf{F} = \mathbf{0}$. Hence, $\mathbf{F}$ is a conservative field and we can be sure that a function f satisfying $\mathbf{F} = \nabla f$ exists.

We have

$$f(x,y,z) = \int F_1\,dx = \int 6xy(\cos z)\,dx = 3x^2y(\cos z) + h_1(y,z),$$
$$f(x,y,z) = \int F_2\,dy = \int 3x^2(\cos z)\,dy = 3x^2y(\cos z) + h_2(x,z),$$
$$f(x,y,z) = \int F_3\,dz = \int -3x^2y(\sin z)\,dz = 3x^2y(\cos z) + h_3(x,y).$$

Thus, f can be taken in the form $f(x,y,z) = 3x^2y(\cos z)$.

3. First, we find

$$\Phi_u(u,v) = (1,1,v), \quad \Phi_v(u,v) = (-1,1,u).$$

The cross product $\Phi_u(1,1) \times \Phi_v(1,1)$ gives a vector which is normal to the given surface at the point $\Phi(1,1) = (0,2,1)$. We calculate

$$\Phi_u(u,v) \times \Phi_v(u,v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & v \\ -1 & 1 & u \end{vmatrix} = (u-v)\mathbf{i} - (u+v)\mathbf{j} + 2\mathbf{k}$$
$$\Phi_u(1,1) \times \Phi_v(1,1) = (0,-2,2).$$

This vector can be taken as a normal vector of the tangent plane at the point $\Phi(1,1)$. Hence, an equation of the tangent plane is

$$0(x-0) - 2(y-2) + 2(z-1) = 0.$$

That finishes the first part.

$$\text{Area} = \int\int_S 1\, dS = \int\int_D 1 \cdot \|\Phi_u(u,v) \times \Phi_v(u,v)\|\, du\, dv$$
$$= \int\int_D \sqrt{(u-v)^2 + (u+v)^2 + 2^2}\, du\, dv = \sqrt{2} \int\int_D \sqrt{u^2+v^2+2}\, du\, dv,$$

where D is the unit disk.

To calculate the latter integral we use the polar coordinates. We let $u = r\cos\theta$ and $v = r\sin\theta$. Then

$$\sqrt{2}\int\int_D \sqrt{u^2+v^2+2}\, du\, dv = \sqrt{2}\int_0^{2\pi}\int_0^1 \sqrt{r^2+2} \cdot \underbrace{r}_{\text{Jacobian}}\, dr\, d\theta = \sqrt{2} \cdot 2\pi \cdot \frac{(r^2+2)^{\frac{3}{2}}}{3}\Big|_0^1$$
$$= \frac{2\sqrt{2}\pi}{3}(3\sqrt{3} - 2\sqrt{2}).$$

4. The given cylinder can be parametrized by

$$\mathbf{T}(\theta, z) = (\cos\theta, \sin\theta, z), \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 1.$$

We calculate

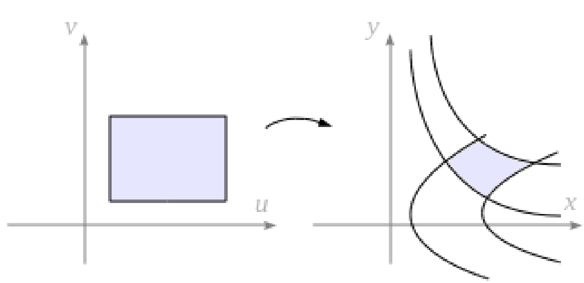
$$\mathbf{T}_\theta(\theta, z) = (-\sin\theta, \cos\theta, 0)$$
$$\mathbf{T}_z(\theta, z) = (0, 0, 1)$$
$$\mathbf{T}_\theta(\theta, z) \times \mathbf{T}_z(\theta, z) = (\cos\theta, \sin\theta, 0).$$

$$\int\int_S \mathbf{F}\, d\mathbf{S} = \int\int_S \mathbf{F} \cdot (\mathbf{T}_\theta \times \mathbf{T}_r)\, dS = \int_0^{2\pi}\int_0^1 (\cos\theta, \sin\theta, -\sin\theta) \cdot (\cos\theta, \sin\theta, 0)\, dz\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 (\cos^2\theta + \sin^2\theta)\, dz\, d\theta = 2\pi.$$

5. The change of variables formula yields

$$\int\int_B x^2 + y^2\, dx\, dy = \int\int_{B^*} \left(x(u,v)^2 + y(u,v)^2\right) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv, \tag{2}$$

where $B^* = [1,4] \times [1,3]$.



We know exactly how u and v depend on x and y, but expressing x and y in terms of u, v seems to be a not so easy problem. It suggests to look for an alternative way of calculating the Jacobian $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. We will use the following property of Jacobians:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}. \tag{3}$$

We find

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2).$$

Then, according to the identity (3),

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2\Big(x(u,v)^2 + y(u,v)^2\Big)}$$

Returning to (2), we obtain

$$\begin{aligned} \int\int_B x^2+y^2\,dx\,dy &= \int\int_{B^*} \Big(x(u,v)^2+y(u,v)^2\Big)\cdot\frac{1}{2\Big(x(u,v)^2+y(u,v)^2\Big)}\,du\,dv \\ &= \frac{1}{2}\int\int_{B^*} 1\,du\,dv = \frac{1}{2}\cdot\text{Area of } B^* = 3. \end{aligned}$$

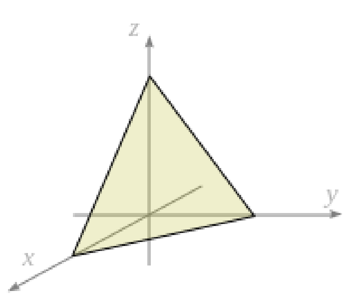
We still do not know the formulas for $x(u,v)$, $y(u,v)$, but nevertheless, we calculated the wanted integral using the change of variables.

6. Short Solution.

By Stokes' Theorem,

$$\int_{C=\partial T} \mathbf{F}\,d\mathbf{S} = \int\int_T (\nabla\times\mathbf{F})\,d\mathbf{S},$$

where $\mathbf{F} = (x+y, 2x-z, y+z)$ and T is the triangle with the vertices $(2,0,0)$, $(0,3,0)$, $(0,0,6)$.



We calculate

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\mathbf{i} - \mathbf{k}.$$

Then

$$\int\int_T (\nabla \times \mathbf{F})\, d\mathbf{S} = \int\int_T (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int\int_T (2,0,1) \cdot \mathbf{n}\, dS, \tag{4}$$

where $\mathbf{n}$ is the unit normal vector to the plane containing the triangle T pointing " upwards" (it is chosen with respect to the right-hand rule).

To find $\mathbf{n}$, let

$$\mathbf{a} = (0,3,0) - (2,0,0) = (-2,3,0)$$
$$\mathbf{b} = (0,0,6) - (2,0,0) = (-2,0,6)$$

and calculate

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix} = 18\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}.$$

Then $\mathbf{n} = \frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|}$. Returning to (4), we obtain

$$\int\int_T (\nabla \times \mathbf{F})\, d\mathbf{S} = \int\int_T (2,0,1) \cdot \frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|}\, dS = \frac{1}{\|\mathbf{a}\times\mathbf{b}\|} \int\int_T (2,0,1)\cdot(18,12,6)\, dS$$
$$= \frac{42}{\|\mathbf{a}\times\mathbf{b}\|} \cdot \int\int_T 1\, dS = \frac{42}{\|\mathbf{a}\times\mathbf{b}\|} \cdot \text{Area of } T.$$

Recall that the area of a triangle with the sides $\mathbf{a}$, $\mathbf{b}$ is equal to $\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$ and conclude that the latter integral is equal to 21.

Long Solution.

Parametrize each side of the triangle, calculate the line integral of $\mathbf{F}$ over each side and sum up the results afterwards.