Math 2374
Spring 2011
Midterm 3
April 21, 2011
Time Limit: 50 minutes

Name (Print): Solutions
Student ID:
Section Number:
Teaching Assistant: Please report errors
Signature:

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

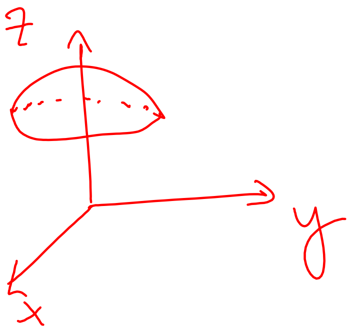
1	25 pts	
2	30 pts	
3	25 pts	
4	30 pts	
5	30 pts	
TOTAL	140 pts	

This page intentionally left blank.

1. (25 points) Let E be the region contained above the plane $z = 2$ and inside the sphere defined by $x^2 + y^2 + z^2 = 16$. Use cylindrical coordinates to set-up an iterated integral to compute

$$\iiint_E x^2 z \, dV.$$

(You do not have to evaluate the iterated integral.)



Bottom face is when $z=2$, top is when $z = \sqrt{16-x^2-y^2}$

These intersect when $2 = \sqrt{16-x^2-y^2}$

i.e. $4 = 16-x^2-y^2$ or $x^2+y^2=12$

radius $2\sqrt{3}$

In cylindrical coords, $x = r\cos\theta$

$y = r\sin\theta$

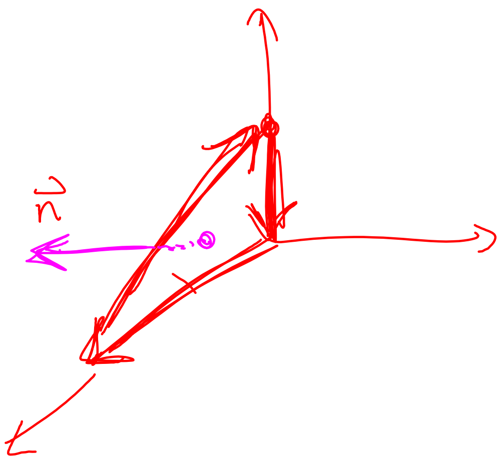
$z = z$

this volume is

$$\int_0^{2\pi}\int_0^{2\sqrt{3}}\int_2^{\sqrt{16-r^2}} r \cdot \, dz \, dr \, d\theta$$

2. (30 points) Let C be the oriented curve (the boundary of a triangle) which moves in straight lines from $(0,0,0)$ to $(2,0,0)$ to $(0,0,1)$ and back to $(0,0,0)$, in that order. Use Stokes' Theorem to calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$, where

$$\mathbf{F}(x,y,z) = \left(-y^2 z, e^{xz}, xy - \sqrt{z^2+1}\right).$$



This triangle is inside the plane $y=0$ and is parametrized by $\Phi(u,v) = (u,0,v)$, $0 \le u \le 1$, $0 \le v \le 1-u$

$T_u = (1,0,0)$ $T_v = (0,0,1)$ $\vec{n} = T_u \times T_v$

$$= \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (0,-1,0)$$

This normal is correctly oriented.

$$\mathrm{Curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ -y^2 z & e^{xz} & xy - \sqrt{z^2+1} \end{vmatrix} = \left(x - xe^{xz}, -y^2 - y, ze^{xz} - 2yz\right)$$

So the integral is

$$\int_0^1 \int_0^{1-u} \left(u - ue^{uv}, 0, ve^{uv}\right) \cdot (0,-1,0)\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} 0 \, dv\, du = 0.$$

3. (25 points) Consider the vector field

$$\mathbf{F}(x, y, z) = \left(y^3 + z\cos(x), 3xy^2 + 1, \sin(x)\right)$$

1. Is there a function $f(x, y, z)$ with $\nabla f = \mathbf{F}$? If so, find f. If not, explain why not.

Check curl(F):

$$\mathrm{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3+z\cos x & 3xy^2+1 & \sin x \end{vmatrix}$$

$= (0, \cos x - \cos x, 3y^2 - 3y^2) = (0,0,0)$ so F is a gradient.

$\frac{\partial f}{\partial x} = y^3 + z\cos x$ $\quad$ $\frac{\partial f}{\partial y} = 3xy^2+1$ $\quad$ $\frac{\partial f}{\partial z} = \sin x$

$f = xy^3 + z\sin x + \underline{c}$ const in x $\quad$ $f = xy^3 + y + \underline{d}$ const in y $\quad$ $f = z\sin x + \underline{e}$ const in z

so $f(x,y,z) = xy^3 + y + z\sin x$

2. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the line segment from $(0, 0, 1)$ to $(0, 2, 3)$.

$$\int_C F \cdot ds = f(0,2,3) - (0,0,1)$$

$$= [0 \cdot 2^3 + 2 + 3\sin(0)] - [0 \cdot 0^3 + 0 + 1 \cdot \sin(0)]$$

$$= 2 - 0 = 2$$

4. (30 points) Give a parametrization for the cone $x^2 + y^2 = z^2$ between $z = 0$ and $z = R$. Use a surface integral to verify that the surface area is $\sqrt{2}\pi R^2$.

Use cylindrical coordinates: $x = r\cos\theta$
$y = r\sin\theta$
$z = z$

In these, $z = r$ describes the cone.

So a parametrization is $r = u$, $\theta = v$
$0 \le u \le R$ $\quad 0 \le v \le 2\pi$

$$\Phi(u,v) = (u\cdot\cos v,\ u\cdot\sin v,\ u)$$

The surface area: $T_u = (\cos v, \sin v, 1)$
$T_v = (-u\cdot\sin v, u\cdot\cos v, 0)$

$$\vec{n} = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-u\cos v, -u\sin v, u)$$

$$\|\vec{n}\| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2}u$$

So the surface area is

$$\int_0^R \int_0^{2\pi} \sqrt{2}u\, dv\, du = \int_0^R 2\pi\sqrt{2}u\, du = \pi\sqrt{2}u^2\Big|_0^R = \pi\sqrt{2}R^2$$

5. (30 points) Let D be the region in the xy-plane bounded by the lines $x+y=1$, $x+y=3$, $y-2x=-1$, $y-2x=1$. Use the transformation $u=x+y$, $v=y-2x$ to compute the double integral

$$\iint_D (x+y)(y-2x)^2\,dx\,dy.$$

(Hint: The reverse transformation is $x=\frac{1}{3}u-\frac{1}{3}v$, $y=\frac{2}{3}u+\frac{1}{3}v$.)

In u,v-coords, this region is described by the lines $u=1$, $u=3$, $v=-1$, $v=1$. The Jacobian is

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{bmatrix}1/3 & -1/3\\ \frac{2}{3} & \frac{1}{3}\end{bmatrix} \qquad |\det J|=\left|\frac{1}{9}+\frac{2}{9}\right|=\frac{1}{3}$$

so the integral becomes

$$\int_{-1}^{1}\int_{1}^{3} u\cdot v^2\cdot\frac{1}{3}\cdot du\,dv$$

$$=\int_{-1}^{1} v^2\cdot\frac{u^2}{6}\Big|_1^3\,dv$$

$$=\int_{-1}^{1} v^2\cdot\left(\frac{9}{6}-\frac{1}{6}\right)dv$$

$$=\int_{-1}^{1}\frac{4}{3}v^2\,dv$$

$$=\frac{4}{9}v^3\Big|_{-1}^{1}=\left(\frac{4}{9}\right)-\left(\frac{-4}{9}\right)=\frac{8}{9}.$$