Math 5345H, Fall 2015
Midterm 2
Due in-class on **Wednesday, November 25**

All questions have equal value.

1. Suppose X is compact, and that $X = \bigcup A_i$ for some subsets A_i. Suppose that for every point p there exists an i such that A_i is a *neighborhood* of p (but A_i is not necessarily open). Show that this cover of X has a finite subcover.

2. Suppose X is a compact metric space and $f : X \to X$ is a function that *strictly decreases* distance: $d(f(x), f(y)) < d(x, y)$ for any $x \neq y$. For any $x_0 \in X$, we can inductively define a sequence $\{x_n\}$ by $x_{n+1} = f(x_n)$. Show that this sequence has a limit x, and that x is the unique point of X satisfying $f(x) = x$. (Hint: Start by showing that it has a convergent subsequence.)

3. Suppose X and Y are spaces and that $f : X \to Y$ is a continuous bijection. Suppose further that

   - X is *locally compact*: every point of X has a compact neighborhood (not necessarily open).
   - Y is *compactly generated*: a subset $C \subset Y$ is closed if and only, for any compact subspace $K \subset Y$, $C \cap K$ is closed in K.

   Is f necessarily a homeomorphism? Prove or give a counterexample.

4. Show explicitly that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the unit n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}.$$

5. Let X be the Sierpinski 2-point space, with points $\{a, b\}$ and open sets $\{\emptyset, \{a\}, X\}$. Describe all four of the functions $X \to X$, and determine which of them are in the subspace $\mathcal{C}(X, X)$ of continuous functions. Determine all the open sets in the compact-open topology on $\mathcal{C}(X, X)$.