Math 5345H, Fall 2015
Final Exam
Due in-class on **Wednesday, December 16**

All questions have equal value.

1. Suppose $U \subset X$ is a subspace. Show that the map $U \to X$ is a covering map if and only if U is both open and closed.

2. Suppose that f and g are continuous functions $X \to Y$, and $H : [0,1] \times X \to Y$ is a homotopy from f to g. Fix a basepoint $x \in X$. Show that there exists a path $\alpha : [0,1] \to Y$, starting at $f(x)$ and ending at $g(x)$, such that

$$f_*(\gamma) = \alpha * g_*(\gamma) * \alpha^{-1}$$

for all $\gamma \in \pi_1(X, x)$.

3. Using the previous exercise, prove the following. Suppose that $f : X \to Y$ and $g : Y \to X$ are continuous functions such that fg is homotopic to id_Y and gf is homotopic to id_X. Then for any basepoint $x \in X$, the map $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.

4. Let $p : Y \to X$ be a covering map where X is path-connected, and $x \in X$. Show that the monodromy operator $y \mapsto y * \gamma$ becomes an *action* of the group $\pi_1(X, x)$ on $p^{-1}(x)$. Show that Y is path-connected if and only if this action has *one* orbit: $p^{-1}(x)$ is nonempty, and for any $y, y' \in p^{-1}(x)$ there exists an element γ such that $y' = y * \gamma$.

5. Let Δ be the triangle $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0, y \leq 2x, 2x + y \leq 2\}$.

   Suppose that $p, q, r \in X$, α is a path from p to q, β is a path from q to r, and γ is a path from p to r. Show that $\alpha * \beta$ is homotopic to γ if and only if there is a continuous function $\sigma : \Delta \to X$ such that

$$\begin{aligned} \sigma(t/2, t) &= \alpha(t) \\ \sigma((1+t)/2, 1-t) &= \beta(t) \\ \sigma(t, 0) &= \gamma(t). \end{aligned}$$

   (HINT: Show that there is an appropriate quotient map from $[0,1] \times [0,1]$ to this triangle.)