Math 5378, Differential Geometry
Practice questions for Test 2

The exam itself will be closed book, no notes.

Note: There are more practice questions appearing here than would appear on an actual exam. The actual exam will have **five** questions, and **two** of them will be off this list.

Solutions will be posted on Monday, April 28.

1. Find all possible trajectories of the vector field $w(x, y) = (-y, x)$ on $\mathbb{R}^2$.

2. If the first fundamental form in coordinates is given by $E = e^u, F = 0, G = e^v$, find a vector field of unit length perpendicular to the vector field $x_u - x_v$.

3. If $f : S_1 \to S_2$ is an isometry between surfaces and $\alpha(s) : (a, b) \to S_1$ is a geodesic parametrized by arc length, show that $f(\alpha(s))$ is also a geodesic parametrized by arc length.

4. Suppose $\mathbf{x}$ is a coordinate chart on a surface, with coefficients E, F, and G of the first fundamental form. Prove the following identities.
$$\begin{aligned} \langle x_{uu}, x_u \rangle &= \frac{1}{2} E_u \\ \langle x_{uu}, x_v \rangle &= F_u - \frac{1}{2} E_v \end{aligned}$$
Use these to show the matrix identity
$$\begin{bmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{bmatrix}$$

5. Prove that the sphere of radius $R > 0$ centered at the origin has constant Gaussian curvature $1/R^2$ and mean curvature $-1/R$.

6. Suppose $(u(s), v(s))$ is a curve in $\mathbb{R}^2$ and $\mathbf{x}$ is a coordinate chart so that $\mathbf{x}(u(s), v(s))$ is a curve parametrized by arc length. Write down the conditions on u and v necessary for this curve to be a geodesic in the surface.

7. Let $\alpha(s) = (f(s), g(s))$ be a curve in $\mathbb{R}^2$ parametrized by arc length, and consider the coordinate chart on the associated surface of revolution given by
$$\mathbf{x}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$
Prove that for any fixed angle θ, the meridian
$$\alpha(s) = (f(s)\cos\theta, f(s)\sin\theta, g(s))$$
is a geodesic parametrized by arc length.

8. Explain the sequence of steps (without calculating anything) taken to derive the Mainardi-Codazzi equations relating Christoffel symbols to e, f, and g from the formulas for x_{uu}, x_{uv}, and x_{vv}.

9. Find the absolute value of the geodesic curvature of the curve $(\cos t\cos\theta, \sin t\cos\theta, \sin\theta)$ on S^2 for any fixed value of θ.

10. On a sphere of radius $R > 0$, suppose that we have a triangle with three geodesic sides, with interior angles θ_1, θ_2, and θ_3. Find the area of the triangle.

11. Show that on a surface of nonpositive curvature, there are no simple closed geodesics that bound simple regions.

12. Calculate the geodesic curvature of the circle $z = h$ on the cone $x^2 + y^2 = z^2$. Explain how the Gauss-Bonnet theorem relates these for different values of h.

13. Calculate the index of the critical point $(0, 0)$ of the vector field
$$w(x, y) = (x^2 - y^2, 2xy)$$
on $\mathbb{R}^2$.