Math 8301, Manifolds and Topology
Homework 4
Due in-class on **Friday, Oct 5**

1. Suppose you are given a simplicial complex with set $\mathcal{V}$ of vertices and set $\mathcal{F}$ of faces. Let X be the space you get by realizing this simplicial complex. For definiteness, we'll let $\mathbb{V}$ be the vector space with basis $\mathcal{V}$, and define

$$X = \bigcup_{U \in \mathcal{F}} \left\{ \sum_{v \in U} t_v \cdot v \,\middle|\, t_v \geq 0, \sum t_v = 1 \right\} \subset \mathbb{V}.$$

   (a) Given an edge $\{a, b\} \in \mathcal{F}$, define a path $p_{a,b}$ from a to b in X.

   (b) Given a triangle $\{a, b, c\} \in \mathcal{F}$, show that there is a homotopy of paths from $p_{a,c}$ to $p_{a,b} \cdot p_{b,c}$.

2. Suppose $f(z)$ is a monic polynomial $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ whose coefficients are complex numbers. Recall $S^1 = \{w \in \mathbb{C} \,|\, |w| = 1\}$. Show that there is a sufficiently large real number $R > 0$ such that

   (a) $f(z) \neq 0$ when $|z| = R$, and

   (b) the resulting function $S^1 \to \mathbb{C} \setminus \{0\}$, given by $w \mapsto f(Rw)$, is homotopic to the map $w \mapsto (Rw)^n$.

3. Suppose M is a n-manifold and that $\gamma : [0, 1] \to M$ is a path in M. Show that there is a homotopic path $\gamma' \sim \gamma$ and an integer n satisfying the following: for all $0 \leq k < n$, there is an open set $U_k \subset M$ and a homeomorphism $\phi_k : U_k \to \mathbb{R}^n$ such that

   - $\gamma'([k/n, (k+1)/n]) \subset U_k$ and
   - the composite function $\phi_k \circ \gamma' : [k/n, (k+1)/n] \to \mathbb{R}^n$ is linear.

4. Let $\mathcal{C}$ be a category, and for any $a, b \in \mathcal{C}$ let $Iso_{\mathcal{C}}(a, b) \subset Hom_{\mathcal{C}}$ be the set of maps $a \to b$ which are isomorphisms in $\mathcal{C}$. Show that there is a *groupoid* $\mathcal{C}^w$ with the same collection of objects as $\mathcal{C}$, but with $Hom_{\mathcal{C}^w}(a, b) = Iso_{\mathcal{C}}(a, b)$.

5. Show that a category with one object is equivalent data to a monoid, and that a groupoid with one object is equivalent data to a group.