Math 8301, Manifolds and Topology
Homework 5
Due in-class on **Friday, Oct 12**

1. (Moment of honesty) For a space X, we'd like to define a groupoid $\Pi_1(X)$ as follows. The objects of $\Pi_1(X)$ are the points of X. The morphisms $Hom_{\Pi_1(X)}(x, y)$ are homotopy classes of paths γ starting at x and ending at y. The composition of morphisms is given by the path composition operation $\gamma \cdot \gamma'$.

   Explain why this, strictly speaking, does not give a definition of a category, and explain how to alter the definition of composition to fix it.

2. If $p : Y \to X$ is a covering space, generalize the action of the fundamental group $\pi_1(X, x)$ on $p^{-1}(x)$ to show that the assignment $x \mapsto p^{-1}(x)$ extends to a functor p^{-1} from $\Pi_1(X)$ to the category of sets. (Of course, this is modulo the necessary alterations from problem 1.)

3. Suppose $p : Y \to X$ and $p' : Y' \to X$ are covering maps, and $\phi : Y \to Y'$ is a homeomorphism such that $p'\phi = p$. Show that the functors p^{-1} and $(p')^{-1}$, from $\Pi_1(X)$ to the category of sets, are naturally isomorphic.

4. Suppose $f(z)$ is a monic polynomial $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ whose coefficients are complex numbers. Define

   $$S = \{z \in \mathbb{C} \mid f'(z) = 0\} \text{ and } T = f(S).$$

   (T is called the set of *singular values* of f.) Use the inverse function theorem, and the fundamental theorem of algebra, to show that the restricted map

   $$f : \mathbb{C} \setminus f^{-1}(T) \to \mathbb{C} \setminus T$$

   is a covering map.

5. Suppose $p : Y \to X$ is a covering map. Here are two plausible-sounding but false statements:

   - If X is an n-manifold, then Y is an n-manifold.
   - If Y is an n-manifold, then X is an n-manifold.

However, a manifold is assumed to be Hausdorff, second countable, and locally homeomorphic to $\mathbb{R}^n$. In each of these two statements, explain which of these three properties succeed or fail to be transported along the covering map. (In at least one case constructing a counterexample turns out to be hard, and so a rough explanation will suffice.)