Math 8302, Manifolds and Topology
Homework 1
Due in-class on Monday, February 4

(Note that these exercises are not identical with the corresponding ones in Lee's text.)

1. Show that two smooth atlases for a manifold M determine the same maximal atlas if and only if their union is a smooth atlas.

2. Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. (Hint: Begin by constructing homeomorphisms from the open unit disc $\mathbb{B}^n$ to itself that are smooth on $\mathbb{B}^n \setminus \{0\}$.)

3. Let $N = (0, \ldots, 0, 1)$ be the "north pole" of $S^n \subset \mathbb{R}^{n+1}$, and let $S = -N$ be the "south pole". Define *stereographic projection* $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by
$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$
Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.
   (a) For any $x \in S^n \setminus \{N\}$, show that $\sigma(x)$ is the point where the line through N and x intersects the plane where $x^{n+1} = 0$.
   (b) Show that σ is bijective, and
$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$
   (c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that these two charts determine a smooth atlas on S^n.

4. An *angle function* on a subset $U \subset S^1 \subset \mathbb{C}$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(p)} = p$ for all $p \in U$. Show that there exists an angle function θ on an open subset $U \subset S^1$ if and only if $U \neq S^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

5. Show that pointwise multiplication turns the set $C^\infty(M)$ of smooth real-valued functions on M into a commutative, associative algebra over $\mathbb{R}$.