Math 8302, Manifolds and Topology II
Homework 6
Due in-class on **Monday, March 25**

Until further notice, all manifolds, maps, vector fields, vector bundles, and other related concepts are smooth unless otherwise specified (or unless you are asked to prove that they are smooth).

1. Suppose $f : E \to M$ is a vector bundle. A (smooth) *section* of f is a map $s : M \to E$ such that $fs = id$. Show that the set of sections of f forms a vector space. Give an example where this vector space is finite dimensional but nonzero.

2. Suppose $f : E \to M$ is a vector bundle of dimension k on M, and U_1 and U_2 are open subsets of of M equipped with *trivializations* of the vector bundle (diffeomorphisms $g_i : f^{-1}U_i \to U_i \times \mathbb{R}^k$ which respect the vector addition). Show that the map
$$g_2 \circ (g_1)^{-1} : (U_1 \cap U_2) \times \mathbb{R}^k \to (U_1 \cap U_2) \times \mathbb{R}^k$$
takes the form of a map $(p, \vec{v}) \mapsto (p, A(p)\vec{v})$ for a smooth map $A : U_1 \cap U_2 \to GL_k(\mathbb{R})$.

3. Suppose $f : E \to M$ is a vector bundle of dimension k, and U is a coordinate chart on M with coordinates $(x^1, \cdots, x^n)$ such that there is a trivialization $f^{-1}(U) \to U \times \mathbb{R}^k$. Construct a set of $n + k$ linearly independent vector fields on $f^{-1}(U)$.

4. Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. Show that the set $U = \{V \in Gr(k, m) \mid V \cap \ker(T) = 0\}$ is an open (but quite possibly empty) subset of $Gr(k, m)$, and that the map $U \to Gr(k, n)$ given by $V \mapsto T(V)$ is smooth.

5. On $\mathbb{R}^3$, there is a distribution generated by the vector fields (CORRECTION) $\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. Suppose we have an *embedding* $S^1 \to \mathbb{R}^3$ which is *tangent* to this distribution: the image of the tangent space of S^1 is always contained inside the distribution. Relate this condition to the formula $dy/dx = z$, and explain (roughly) how such an embedding might be determined the picture given by projecting it onto the xy-plane.