Math 8302, Manifolds and Topology II
Homework 1
Due in-class on **Monday, February 2**

Recall for these exercises that, for subspaces $K \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^m$, a map $f : K \to L$ is smooth if, for all $p \in K$, there exists an open neighborhood $p \in U \subset \mathbb{R}^n$ and a smooth function $g : U \to \mathbb{R}^m$ such that $f|_{U\cap K} = g|_{U\cap K}$.

1. Suppose $K \subset \mathbb{R}^n$ and $p \in K$. Define $T_p(K)$ to be the set of vectors $\vec{v} \in \mathbb{R}^n$ such that there exists an $\epsilon > 0$ and a smooth function $c : (-\epsilon, \epsilon) \to \mathbb{R}^n$ such that $c(0) = p$, $c([0, \epsilon)) \subset K$, and $c'(0) = \vec{v}$. We will call this set the *tangent cone* of K at p.

   Show that if $\vec{v} \in T_p(K)$ and $r \geq 0$, then $r\vec{v} \in T_p(K)$.

2. Describe the tangent cones of the following subspaces of $\mathbb{R}^2$ at $(0, 0)$.

   (a) The parabola $\{(x, y) | y = x^2\}$.

   (b) For any $m > 0$, the ray $R_m = \{(x, y)|y = mx, x \geq 0, y \geq 0\}$.

   (c) The set $\bigcup_{m>0} R_m$. (Careful!)

3. Show that a smooth function $f : K \to L$ determines a well-defined function $df_p : T_p(K) \to T_{f(p)}(L)$ such that there exists a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ with $df_p = A|_{T_p(K)}$. Show also that $d(g \circ f)_p = dg_{f(p)} \circ df_p$ and $d(id)_p = id_{T_p(K)}$.

4. Show that the diamond $\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$ and the circle $S^1 \subset \mathbb{R}^2$ are not diffeomorphic.

5. (a) Show that, for all real numbers α, $\lim_{y\to\infty} y^\alpha e^{-y} = 0$.

   (b) Use this, show that for all real numbers β, $\lim_{x\to 0} x^\beta e^{-1/x^2} = 0$.

   (c) Show that the function

   $$\phi(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   is a smooth function from $\mathbb{R}$ to $\mathbb{R}$.