Math 8302, Manifolds and Topology II
Homework 2
Due in-class on **Monday, February 9**

1. We mentioned, but did not show in class, that for a smooth map $f : M \to N$, the set of regular points is an open subset of M. Give a proof of this. (You may assume that "row reduction works": an $m \times n$ matrix A is surjective if and only if there is a set $\{k_1 < k_2 < \ldots < k_m\}$ of columns of A so that the resulting submatrix $(A_{i,k_j})_{i,j}$ is invertible.)

2. Determine the set of singular values of the function $f : S^2 \to \mathbb{R}^2$ given by $f(x, y, z) = (2x + y, 2x - y)$.

3. Suppose $M \subset \mathbb{R}^n$ is a smooth manifold of dimension k. Define
$$T(M) = \{(p, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n \mid p \in M, \vec{v} \in T_p(M)\}$$
Show that $T(M)$ is a smooth manifold of dimension $2k$, and that the projection $\pi : T(M) \to M$ given by $\pi(p, \vec{v}) = p$ is a smooth map.

4. Suppose that $f : M \to N$ is a smooth map. Show that the function $df : T(M) \to T(N)$ given by $df(p, \vec{v}) = (f(p), df_p(\vec{v}))$ is a smooth map, and that $d(g \circ f) = dg \circ df$.

5. On the last homework, you constructed a smooth function f on $\mathbb{R}$ which was positive on $(0, \infty)$ and zero otherwise. Use this to construct smooth functions $\mathbb{R} \to \mathbb{R}$ of the following types.

   (a) A smooth function g which is positive on $(-1, 1)$ and zero otherwise.

   (b) A smooth function h with
$$\begin{aligned} h(x) &= 0 & x &\le -1, \\ h(x) &= 1 & x &\ge 1, \\ 0 < h(x) &< 1 & |x| &< 1. \end{aligned}$$

   (c) A smooth function k with
$$\begin{aligned} k(x) &= 1 & |x| &\le 1, \\ k(x) &= 0 & |x| &\ge 2, \\ 0 < k(x) &< 1 & 1 < |x| &< 2. \end{aligned}$$