Math 8302, Manifolds and Topology II
Homework 3
Due in-class on **Monday, February 16**

In the following, an *abstract* m-dimensional smooth manifold is an n-dimensional topological manifold M (Hausdorff, second countable, and every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^m$) together with a *coordinate atlas*: a set

$$A = \{\varphi_\alpha : U_\alpha \to V_\alpha\}_\alpha$$

of "charts": homeomorphisms from open subsets U_α of M to open subsets V_α of $\mathbb{R}^m$. In addition, we require that

- the U_α cover M in the sense that $M = \bigcup U_\alpha$, and
- the charts are *compatible* in the sense that the maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

  are smooth for all choices of α and β.

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1. If M is an abstract smooth manifold, a function $M \to \mathbb{R}^k$ is *smooth at* p if it is smooth in coordinates: there exists a coordinate chart $\varphi_\alpha : U_\alpha \to V_\alpha$ in the atlas with $p \in U_\alpha$ such that the function $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}^k$ is smooth at p. Show that this is independent of the choice of coordinate chart.

2. Show that an abstract smooth manifold M has an exhaustion by compact sets: there exists a sequence of compact subspaces $K_1 \subset K_2 \subset \cdots$ such that $M = \bigcup K_i$. (Hint: Second countability is critical here.)

3. Suppose M is an abstract smooth manifold, $K \subset U \subset M$ an inclusion of a compact subset into an open subset. Suppose we have a smooth function $f : U \to \mathbb{R}^k$. Show that there exists an smooth function $g : M \to \mathbb{R}^k$ such that $f|_K = g|_K$ and $g|_{M \setminus U} = 0$.

4. Show that an abstract smooth M has a smooth function $T : M \to \mathbb{R}$ such that $T^{-1}((-\infty, r])$ is compact for any $r \in \mathbb{R}$. (Hint: Use an exhaustion by compact sets.)

5. Suppose $f(x)$ is a smooth function $\mathbb{R} \to \mathbb{R}$ such that

   (a) $f(x) = 0$ for $x \neq (-1, 1)$,

   (b) $f(0) = 1$,

   (c) $f(x) = f(-x)$,

   (d) $f'(x) < 0$ for $x \in (0, 1/2)$.

   Suppose that we are given, for each integer n, a real number a_n such that $a_n \neq a_{n+1}$. Construct a smooth function $g(x) : \mathbb{R} \to \mathbb{R}$ whose set of singular points is $\mathbb{Z}$ and such that $f(n) = a_n$.