Math 8302, Manifolds and Topology II
Homework 4
Due in-class on **Monday, March 2**

1. Let $M = (0, \infty) \times (0, \infty) \subset \mathbb{R}^2$, and define new coordinates on M by $u = xy$, $v = x/y$.

   (a) Convert the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ into (u, v)-coordinates.

   (b) Convert the vector fields $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ into (x, y)-coordinates.

   (c) Find the maximal submanifold of $M \times \mathbb{R}$ on which a flow $\Theta(x, y, t)$ for $\frac{\partial}{\partial u}$ is defined.

2. Suppose M and N are manifolds, $f : M \to N$ is an immersion, and X is a vector field on N with the following property: *For all $p \in M$, the vector $X(f(p))$ is in the image of the map $df_p : T_p(M) \to T_{f(p)}(N)$.*

   Show that this determines a *smooth* vector field $\tilde{X}$ on M such that $df_p(\tilde{X}(p)) = X(f(p))$ for all $p \in M$. (Hint: Show first that there's a unique definition of $\tilde{X}$ and then verify that it's smooth.)

3. Fix a vector $\vec{v}$ in $\mathbb{R}^3$, and consider the function $X_{\vec{v}}$ which sends a point p of the smooth manifold $\mathbb{R}^3$ to $X_{\vec{v}}(p) = \vec{v} \times \overrightarrow{0p}$, the vector cross product of $\vec{v}$ with the vector from the origin to p. First, show that this defines a smooth vector field on $\mathbb{R}^3$. Second, if $\vec{w}$ is another vector, determine the Lie bracket $[X_{\vec{v}}, X_{\vec{w}}]$ of the vector field $X_{\vec{v}}$ and the vector field $X_{\vec{w}}$.

4. Suppose M is n-dimensional with a chosen point p, and $X_1, \ldots, X_n$ are vector fields on M so that $\{X_i(p)\}$ is a basis of the tangent space $T_p(M)$.

   For each i, let $\Theta_i : U_i \to M$ be a flow for the vector field X_i (defined on some open set U_i with $M \times \{0\} \subset U_i \subset M \times \mathbb{R}$).

   Inductively define functions f_j on an open neighborhood of $0 \in \mathbb{R}^j$ as follows. The function $f_0 : \mathbb{R}^0 \to M$ sends 0 to p. Then

   $$f_j(t^1, \ldots, t^j) = \Theta_j(f_{j-1}(t^1, \ldots, t^{j-1}), t^j).$$

   Consider the maps $(df_j)_0 : \mathbb{R}^j \to T_p(M)$. Show that the expression of this in the basis $\{X_i(p)\}$ of $T_p(M)$ is a matrix with ones on the diagonal and zeroes elsewhere. Explain why the restriction of the map f_n gives a diffeomorphism from an open neighborhood of $0 \in \mathbb{R}^n$ to an open neighborhood of $p \in M$.

5. In the situation of the previous problem, suppose that the Lie brackets $[X_i, X_j]$ are all zero. Show that the map $(f_n)^{-1}$ gives a coordinate chart in a neighborhood of p so that, in these coordinates, $X_i = \frac{\partial}{\partial t^i}$.